# SM015/2
# PSPM 1

# 2019/2020


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# Questions

## SECTION A [45 marks]

***This section consists of 5 questions. Answer all questions.***

1. Polynomial $P(x) = 12x^3 - px^2 - qx + 8$ is divisible by $3x^2 - 7x + 4$. Find the values of $p$ and $q$. Hence, factorise $P(x)$ completely.

**[8 marks]**

2. a) Show that $cosec\,\theta - \cot\theta$ can be simplified as $\tan\frac{\theta}{2}$. **[5 marks]**

b) Hence, find the values of $\mathbf{\tan 15°}$ and $\boldsymbol{sec^2 15°}$. Give your answer in the form $\boldsymbol{a + b\sqrt{c}}$, where a, b and c are integers.

**[5 marks]**

3. a) A function $f(x)$ is defined as

$$\boldsymbol{f(x) = \begin{cases} \dfrac{9 - x^2}{x - 3}, & x < 3 \\ -6, & x = 3 \\ \dfrac{162x - 54x^2}{x^3 - 27}, & x > 3 \end{cases}}$$

Determine whether $\boldsymbol{f(x)}$ is continuous at $\boldsymbol{x = 3}$.

**[7 mark]**

b) Find $\displaystyle\lim_{x\to\infty} \frac{5x^3+4x-9}{7-3x^3}$. **[3 mark]**

4. a) Given $y = x^2e^{-3x}$, find $\frac{d^2y}{dx^2}$. Give your answer in the simplest form. **[5 mark]**

b) Given $\boldsymbol{x^2y = \sin(2x^2 + \pi)}$.

Show that $\boldsymbol{2y(1 + 8x^4) + 4x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 4\cos(2x^2 + \pi)}$.

**[7 marks]**

5. Given $f(x) = x^2 - \frac{6}{x}$, where $x \neq 0$. Find the coordinate(s) of stationary point and determine whether it is a relative maximum or a relative minimum. Give the coordinates correct to 3 decimal places.

**[5 marks]**

## SECTION B [25 marks]

***This section consists of 2 question. Answer all questions.***

1. The function $g(x)$ is defined by

$$\boldsymbol{g(x) = \begin{cases} m^2 - 8, & x \leq 2 \\ \dfrac{x-2}{\sqrt{2x}-2}, & 2 < x \leq 8 \\ \dfrac{|8-x|}{x-8} + k, & x > 8 \end{cases}}$$

Where $\boldsymbol{m}$ and $\boldsymbol{k}$ is constant.

a. If $\lim_{x \to 2} g(x)$ exists, find the values of $m$. **[5 marks]**

b. Find the values of $k$ such that $g(x)$ is discontinuous at $x = 8$. **[3 marks]**

2. Given a right circular cylinder with height $2h$ and radius $r = \sqrt{a^2 - h^2}$, which is inscribed in a sphere of fixed radius a.

a. Show that the volume of the cylinder is $V = 2\pi(a^2 - h^2)h$. **[5 marks]**

b. From part 2(a), find the maximum value of the volume, $V$ in terms of $\pi$ as the height, $h$ varies. Hence, state the value of the volume if $a = 3$. **[9 marks]**

c. Show that the ratio of the volume of the sphere to the maximum volume of the cylinder is $\boldsymbol{\sqrt{3} : 1}$.

**[3 marks]**

**END OF QUESTION PAPER**

# Question A1

1. Polynomial $P(x) = 12x^3 - px^2 - qx + 8$ is divisible by $3x^2 - 7x + 4$. Find the values of $p$ and $q$. Hence, factorise $P(x)$ completely.

**[8 marks]**

## SOLUTION

$$P(x) = 12x^3 - px^2 - qx + 8$$

$$D(x) = 3x^2 - 7x + 4 = (3x - 4)(x - 1)$$

$$P(1) = 0$$

$$P\left(\frac{4}{3}\right) = 0$$

$$P(1) = 12(1)^3 - p(1)^2 - q(1) + 8 = 0$$

$$p + q = 20$$

$$p = 20 - q \qquad \text{................ } (1)$$

$$P\left(\frac{4}{3}\right) = 12\left(\frac{4}{3}\right)^3 - p\left(\frac{4}{3}\right)^2 - q\left(\frac{4}{3}\right) + 8 = 0$$

$$\frac{768}{27} - \frac{16}{9}p - \frac{4}{3}q + 8 = 0$$

$$768 - 48p - 36q + 216 = 0$$

$$48p + 36q = 984 \qquad \text{................ } (2)$$

Substitute (1) into (2)

$$48(20 - q) + 36q = 984$$

$$960 - 48q + 36q = 984$$

$$-12q = 24$$

$$q = -2$$

$$p = 22$$

$$P(x) = 12x^3 - 22x^2 + 2x + 8$$

$$
\begin{array}{r l}
 & 4x + 2 \\
3x^2 - 7x + 4 \,\big) & 12x^3 - 22x^2 + 2x + 8 \\
 & 12x^3 - 28x^2 + 16x \\
\hline
 & 6x^2 - 14x + 8 \\
 & 6x^2 - 14x + 8 \\
\hline
\end{array}
$$

$P(x) = 12x^3 - 22x^2 + 2x + 8$

$= (3x^2 - 7x + 4)(4x + 2)$

$= (3x - 4)(x - 1)\,(4x + 2)$

$= 2(3x - 4)(x - 1)\,(2x + 1)$

**Alternative Method:**

$P(x) = 12x^3 - px^2 - qx + 8$

$D(x) = 3x^2 - 7x + 4 = (3x - 4)(x - 1)$

$P(1) = 0$ ➔ $(x - 1) is\ a\ factor\ of\ P(x)$

$P\left(\frac{4}{3}\right) = 0$ ➔ $(3x - 4)\ is\ a\ factor\ of\ P(x)$

$P(x) = Q(x)D(x)$

$12x^3 - px^2 - qx + 8 = (3x - 4)(x - 1)(ax + b)$

$12x^3 - px^2 - qx + 8 = (3x^2 - 3x - 4x + 4)(ax + b)$

$12x^3 - px^2 - qx + 8 = (3x^2 - 7x + 4)(ax + b)$

$12x^3 - px^2 - qx + 8 = 3ax^3 + 3bx^2 - 7ax^2 - 7bx + 4ax + 4b$

$12x^3 - px^2 - qx + 8 = 3ax^3 + (3b - 7a)x^2 + (4a - 7b)x + 4b$

**Compare the coefficient:**

$x^3$: $3a = 12$

$a = 4$

$x^0$: $8 = 4b$

$b = 2$

$x^2$: $-p = 3b - 7a$

$-p = 3(2) - 7(4)$

$p = 22$

$x$: $-q = 4a - 7b$

$-q = 4(4) - 7(2)$

$q = -2$

$p = 22; q = -2$

$P(x) = (3x - 4)(x - 1)\ (4x + 2)$

$= 2(3x - 4)(x - 1)\ (2x + 1)$

# Question A2

2. a) Show that $cosec\ \theta - \cot\theta$ can be simplified as $\tan\frac{\theta}{2}$. **[5 marks]**

b) Hence, find the values of $\mathbf{tan\ 15°}$ and $\boldsymbol{sec^2 15°}$. Give your answer in the form of $\boldsymbol{a + b\sqrt{c}}$, where a, b and c are integers. **[5 marks]**

## SOLUTION

a) $cosec\ \theta - cot\ \theta = \frac{1}{sin\,\theta} - \frac{cos\,\theta}{sin\,\theta}$

$$= \frac{1 - cos\,\theta}{sin\,\theta}$$

$$= \frac{1 - \left(1 - 2sin^2\frac{\theta}{2}\right)}{2\,sin\frac{\theta}{2}cos\frac{\theta}{2}}$$

$$= \frac{2sin^2\frac{\theta}{2}}{2\,sin\frac{\theta}{2}cos\frac{\theta}{2}}$$

$$= \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}$$

$$= tan\frac{\theta}{2}$$

b) $\tan\frac{\theta}{2} = cosec\ \theta - cot\ \theta$

$$tan\,15° = tan\left(\frac{30°}{2}\right)$$

$$= cosec\,30° - cot\,30°$$

$$= \frac{1}{sin\,30°} - \frac{cos\,30°}{sin\,30°}$$

$$= \frac{1}{\frac{1}{2}} - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= 2 - \sqrt{3}$$

$$\boldsymbol{sec^2\theta = \tan^2\theta + 1}$$

$$\begin{aligned}\boldsymbol{sec^2 15^\circ} &= \boldsymbol{\tan^2 15^\circ + 1}\\ &= \boldsymbol{(tan\,15^\circ)^2 + 1}\\ &= \boldsymbol{\left(2-\sqrt{3}\right)^2 + 1}\\ &= \boldsymbol{\left(4+3-4\sqrt{3}\right) + 1}\\ &= \boldsymbol{8-4\sqrt{3}}\\ &= \boldsymbol{4\left(2-\sqrt{3}\right)}\end{aligned}$$

# Question A3

3. a) A function $f(x)$ is defined as

$$f(x)=\begin{cases}\dfrac{9-x^2}{x-3}, & x<3\\ -6, & x=3\\ \dfrac{162x-54x^2}{x^3-27}, & x>3\end{cases}$$

Determine whether $f(x)$ is continuous at $x=3$.

**[7 mark]**

b) Find $\lim_{x\to\infty}\frac{5x^3+4x-9}{7-3x^3}$. **[3 mark]**

## SOLUTION

a) $f(x)=\begin{cases}\frac{9-x^2}{x-3}, & x<3\\ -6, & x=3\\ \frac{162x-54x^2}{x^3-27}, & x>3\end{cases}$

At $x=3$

$f(3)=-6$

$$\lim_{x\to3^-}f(x)=\lim_{x\to3^-}\left(\frac{9-x^2}{x-3}\right)$$

$$=\lim_{x\to3^-}\left[\frac{(3+x)(3-x)}{x-3}\right]$$

$$=\lim_{x\to3^-}\left[\frac{-(3+x)(x-3)}{x-3}\right]$$

$$=\lim_{x\to3^-}[-(3+x)]$$

$$=-(3+3)$$

$$=-6$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}\left(\frac{162x-54x^2}{x^3-27}\right)$$

$$= \lim_{x\to 3^+}\left[\frac{-54x(x-3)}{(x-3)(x^2+3x+9)}\right]$$

$$= \lim_{x\to 3^+}\left[\frac{-54x}{(x^2+3x+9)}\right]$$

$$= \frac{-54(3)}{3^2+3(3)+9}$$

$$= \frac{-162}{27}$$

$$= -6$$

$$\begin{array}{r|l} & x^2+3x+9 \\ \hline x-3 & x^3-27 \\ & x^3-3x^2 \\ \hline & 3x^2-27 \\ & 3x^2-9x \\ \hline & 9x-27 \\ & 9x-27 \\ \hline \end{array}$$

$$\lim_{x\to 3} f(x) = -6$$

$\boldsymbol{Since\ f(3) = \lim_{x\to 3} f(x) = -6\,, therefore}$

$\boldsymbol{f(x) is\ continuous\ at\ x = 3}$

b) $\lim_{x\to\infty}\frac{5x^3+4x-9}{7-3x^3} = \lim_{x\to\infty}\frac{5x^3+4x-9}{7-3x^3}$

$$= \lim_{x\to\infty}\frac{\frac{5x^3+4x-9}{x^3}}{\frac{7-3x^3}{x^3}}$$

$$= \lim_{x\to\infty}\frac{5+\frac{4}{x^2}-\frac{9}{x^3}}{\frac{7}{x^3}-3}$$

$$= \frac{5+0-0}{0-3}$$

$$= -\frac{5}{3}$$

# Question A4

4. a) Given $y = x^2 e^{-3x}$, find $\frac{d^2y}{dx^2}$. Give your answer in the simplest form. **[5 mark]**

b) Given $x^2 y = \sin(2x^2 + \pi)$.

Show that $2y(1 + 8x^4) + 4x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = 4\cos(2x^2 + \pi)$. **[7 marks]**

## SOLUTION

a) $y = x^2 e^{-3x}$

$u = x^2 \qquad v = e^{-3x}$

$u' = 2x \qquad v' = -3e^{-3x}$

$$\frac{dy}{dx} = uv' + vu'$$

$$= (x^2)(-3e^{-3x}) + (e^{-3x})(2x)$$

$$= e^{-3x}[-3x^2 + 2x]$$

$u = -3x^2 + 2x \qquad v = e^{-3x}$

$u' = -6x + 2 \qquad v' = -3e^{-3x}$

$$\frac{d^2y}{dx^2} = (-3x^2 + 2x)(-3e^{-3x}) + (e^{-3x})(-6x + 2)$$

$$= e^{-3x}[(-3)(-3x^2 + 2x) + (-6x + 2)]$$

$$= e^{-3x}[9x^2 - 6x - 6x + 2]$$

$$= e^{-3x}[9x^2 - 12x + 2]$$

b) $x^2 y = \sin(2x^2 + \pi)$

$$x^2\frac{dy}{dx} + 2xy = 4x\cos(2x^2 + \pi)$$

$$\left[x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}\right]+\left[2x\frac{dy}{dx}+2y\right]=\left[-4x(4x)\sin(2x^2+\pi)+4\cos(2x^2+\pi)\right]$$

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2x\frac{dy}{dx}+2y=-16x^2\sin(2x^2+\pi)+4\cos(2x^2+\pi)$$

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=-16x^2(x^2y)+4\cos(2x^2+\pi)$$

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y+16x^2(x^2y)=4\cos(2x^2+\pi)$$

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y(1+8x^4)=4\cos(2x^2+\pi)$$

$$2y(1+8x^4)+4x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}=4\cos(2x^2+\pi)$$

Given:
$x^2y=\sin(2x^2+\pi)$

# Question A5

5. Given $f(x) = x^2 - \frac{6}{x}$, where $x \neq 0$. Find the coordinate(s) of stationary point and determine whether it is a relative maximum or a relative minimum. Give the coordinates correct to 3 decimal places.

**[5 marks]**

## SOLUTION

$$f(x) = x^2 - \frac{6}{x}$$

$$= x^2 - 6x^{-1}$$

$$f'(x) = 2x + 6x^{-2}$$

$$= 2x + \frac{6}{x^2}$$

$$Let\ f'(x) = 0$$

$$2x + \frac{6}{x^2} = 0$$

$$2x^3 + 6 = 0$$

$$2x^3 = -6$$

$$x^3 = -3.$$

$$x = -1.442$$

$$When\ x = -1.442,$$

$$f(x) = (-1.442)^2 - \frac{6}{-1.442}$$

$$= 6.241$$

$$The\ stationary\ point = (-1.442, 6.241)$$

$$\frac{d^2y}{dx^2} = 2 - 12x^{-3}$$

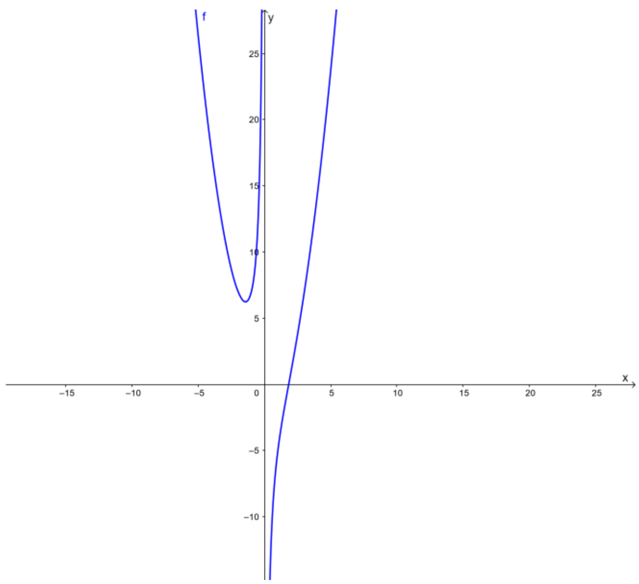
$$= 2 - \frac{12}{x^3}$$

$$When\ x = -1.44$$

$$\frac{d^2y}{dx^2} = 2 - \frac{12}{(-1.44)^3} = 6.002 > 0\ (Minimum)$$

$$\therefore The\ point\ (-1.44, 6.241)\ is\ relative\ minimum.$$

Graph: (Only for illustration purpose)



# Question B1

1. The function $g(x)$ is defined by

$$g(x)=\begin{cases} m^2-8, & x\le 2\\ \dfrac{x-2}{\sqrt{2x}-2}, & 2<x\le 8\\ \dfrac{|8-x|}{x-8}+k, & x>8\end{cases}$$

Where $m$ and $k$ is constant.

a. If $\lim_{x\to 2} g(x)$ exists, find the values of $m$. **[5 marks]**

b. Find the values of $k$ such that $g(x)$ is discontinuous at $x=8$. **[3 marks]**

## SOLUTION

$$g(x)=\begin{cases} m^2-8, & x\le 2\\ \dfrac{x-2}{\sqrt{2x}-2}, & 2<x\le 8\\ \dfrac{|8-x|}{x-8}+k, & x>8\end{cases}$$

a) $\lim_{x\to 2} g(x)$

$$\lim_{x\to 2^-} g(x)=\lim_{x\to 2^+} g(x)$$

$$\lim_{x\to 2^-} m^2-8=\lim_{x\to 2^+}\frac{x-2}{\sqrt{2x}-2}$$

$$m^2-8=\lim_{x\to 2^+}\frac{x-2}{\sqrt{2x}-2}\,\frac{\sqrt{2x}+2}{\sqrt{2x}+2}$$

$$m^2-8=\lim_{x\to 2^+}\frac{(x-2)\left(\sqrt{2x}+2\right)}{2x-4}$$

$$m^2-8=\lim_{x\to 2^+}\frac{(x-2)\left(\sqrt{2x}+2\right)}{2(x-2)}$$

$$m^2-8=\lim_{x\to 2^+}\frac{\sqrt{2x}+2}{2}$$

$$m^2 - 8 = \frac{\sqrt{2(2)}+2}{2}$$

$$m^2 - 8 = 2$$

$$m^2 = 10$$

$$m = \pm\sqrt{10}$$

b) $g(x)$ is discontinuous at $x = 8$

$$f(8) = \frac{8-2}{\sqrt{2(8)}-2}$$

$$f(8) = \frac{6}{2} = 3$$

$$\lim_{x\to 8^+} \frac{|8-x|}{x-8} + k = \lim_{x\to 8^+} \frac{-8+x}{x-8} + k$$

$$= \lim_{x\to 8^+} 1 + k$$

$$= 1 + k$$

$$|8-x| = \begin{cases} 8-x, & 8-x \geq 0 \\ -(8-x), & 8-x < 0 \end{cases}$$

$$= \begin{cases} 8-x, & x \leq 8 \\ -8+x, & x > 8 \end{cases}$$

*Since $g(x)$ is discontinuous at $x = 8$*

$$1 + k \neq 3$$

$$k \neq 2$$

$$k \in \mathbb{R}\backslash\{2\}$$

# Question B2

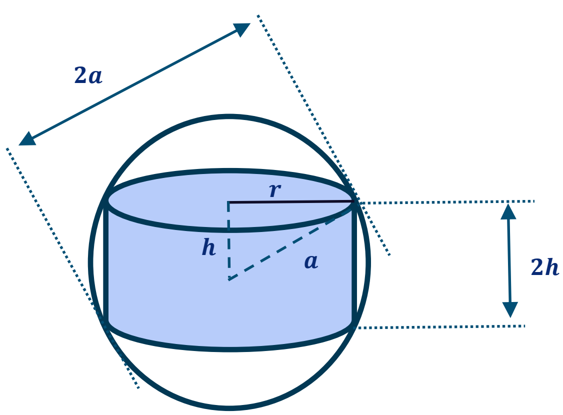
2. Given a right circular cylinder with height $2h$ and radius $r = \sqrt{a^2 - h^2}$, which is inscribed in a sphere of fixed radius a.

   a. Show that the volume of the cylinder is $V = 2\pi(a^2 - h^2)h$. **[5 marks]**

   b. From part 2(a), find the maximum value of the volume, $V$ in terms of $\pi$ as the height, $h$ varies. Hence, state the value of the volume if $a = 3$. **[9 marks]**

   c. Show that the ratio of the volume of the sphere to the maximum volume of the cylinder is $\sqrt{3}:1$. **[3 marks]**

## SOLUTION




a) $r^2 + h^2 = a^2$

$$r^2 = a^2 - h^2$$

$$r = \sqrt{a^2 - h^2}$$

$$Volume\ of\ cylinder = \pi r^2(2h)$$

$$= 2\pi\left(\sqrt{a^2 - h^2}\right)^2 h$$

$$= 2\pi(a^2 - h^2)h$$

b) $V = 2\pi(a^2 - h^2)h$

$= 2\pi a^2 h - 2\pi h^3$

$\frac{dV}{dh} = 2\pi a^2 - 6\pi h^2$

$Let\ \frac{dV}{dh} = 0;$

$2\pi a^2 - 6\pi h^2 = 0$

$2\pi(a^2 - 3h^2) = 0$

$a^2 - 3h^2 = 0$

$3h^2 = a^2$

$h^2 = \frac{a^2}{3}$

$h = \frac{a}{\sqrt{3}}$

$\frac{dV}{dh} = 2\pi a^2 - 6\pi h^2$

$\frac{d^2V}{dh^2} = -12\pi h < 0 \qquad (as\ h > 0)$

$When\ a = 3$

$h^2 = \frac{3^2}{3} = 3$

$h = \sqrt{3}$

$r^2 = a^2 - h^2$

$= 3^2 - 3$

$= 6$

$V_{max} = 2\pi(a^2 - h^2)h$

$= 2\pi(3h^2 - h^2)h$

$= 2\pi(2h^3)$

$= 4\pi h^3$

$= 4\pi\sqrt{3}^3$

$$= 12\sqrt{3}\pi$$

c) Volume of sphere $V = \frac{4}{3}\pi r^3$

$$V_S = \frac{4}{3}\pi a^3 = \frac{4\pi a^3}{3}$$

$$V_{max} = 4\pi h^3 = 4\pi\left(\frac{a}{\sqrt{3}}\right)^3 = \frac{4\pi a^3}{3\sqrt{3}}$$

$$\frac{V_S}{V_{max}} = \frac{\frac{4\pi a^3}{3}}{\frac{4\pi a^3}{3\sqrt{3}}}$$

$$= \frac{4\pi a^3}{3}\left(\frac{3\sqrt{3}}{4\pi a^3}\right)$$

$$= \frac{\sqrt{3}}{1}$$

$$V_S : V_{max} = \sqrt{3} : 1$$