

SM025/2

Matriculation Programme Examination

Semester 2

Session 2018/2019

- 1. a) Evaluate $\int x(1-x)^8 dx$ by using a suitable substitution.
 - b) Find the area of the region bounded by the curve $y = x \cos x$ and x axis between x = 0 and $x = \frac{\pi}{3}$. Give your solution in term of π .
- 2. Solve $\frac{dy}{dx} + 2(x+1)y^2 = 0$, given that y = 1 when x = 1. Express y in terms of x.
- 3. Use Newton-Raphson method to solve the equation $e^x x 2 = 0$ correct to four decimal places by taking $x_1 = 1$ as the first approximation.
- 4. An ellipse $Ax^2 + y^2 + Bx + Cy + 1 = 0$ passes through points (0,1), (1,-1) and (2,1).
 - a) Find the equation of the ellipse in the standard form. Hence, state the centre and vertices of the ellipse.
 - b) Find the foci of th ellipse.
 - c) Sketch the graph of the ellipse.
- 5. The line L_1 and L_2 passes through the point R(2, 4, -3) and S(8, -5, 9) in the direction of $2\mathbf{i} 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} 2\mathbf{j} + 3\mathbf{k}$, respectively.
 - a) State the equations for lines L_1 and L_2 in the vector form. Hence, calculate the acute angle between the lines L_1 and L_2 .
 - b) Find the equation of plane containing the line L_1 and the point (7, -3, 5) in the Cartesian form.
 - c) Determine whether the line L_2 is parallel to the plane x + 5y + 3z = 5.
- 6. **TABLE 1** shows the frequency distribution of the diameter of 100 pebbles which are measured to the nearest millimeter(mm).

TABLE 1

Diameter (mm)	Frequency
10 – 14	22
15 – 19	20
20 – 24	25
25 – 29	15
30 - 34	18

Calculate the

- a) mean.
- b) mode.
- c) median.

- 7. A committee of 5 is to be selected from a group of 7 men and 6 women. How many different committees could be formed if
 - a) there is no woman in the committee?
 - b) a particular man must be in the committee and the remaining has equal number of men and women?
 - c) at least 3 men are in the committee?
- 8. The probabilities of events X and Y are given as $P(X) = \frac{3}{5}$, $P(X'|Y) = \frac{31}{45}$ and $P(X \cap Y) = \frac{2}{25}$.
 - a) Show that $P(Y) = \frac{9}{35}$
 - b) Find $P(X \cup Y')$.
- 9. A discrete random variable X has the probability distribution function

$$f(x) = \begin{cases} \frac{x+1}{16}, & x = 2, 3, 4\\ kx, & x = 6, 8\\ 0, & otherwise \end{cases}$$

- a) Show that $k = \frac{1}{56}$
- b) Hence, calculate $P(3 \le X < 8)$.
- c) Determine the values of E(X) and Var(X). Thus, evaluate $Var(\sqrt{3}X 1)$
- 10. The number of batteries sold at a service center on any particular day follows a Poisson distribution with mean λ .
 - a) If the probability of selling exactly 4 batteries divided by the probability of selling exactly 2 batteries is $\frac{225}{12}$, show that $\lambda=15$.
 - b) On any particular day, calculate the probability that the service center sells between 5 and 14 batteries.
 - c) Given that the probability of selling less than k batteries on any particular day is 0.917, find the value of k.
 - d) Find the probability that exactly 40 batteries are sold in 2 working days. Give your answer in four decimal places.

END OF QUESTION PAPER

- 1. a) Evaluate $\int x(1-x)^8 dx$ by using a suitable substitution.
 - b) Find the area of the region bounded by the curve $y = x \cos x$ and x axis between x = 0 and $x = \frac{\pi}{2}$. Give your solution in term of π .

(a)
$$\int x(1-x)^8 dx$$

$$u = 1-x \implies x = 1-u$$

$$\frac{du}{dx} = -1$$

$$dx = -du$$

$$\int x(1-x)^8 dx = \int (1-u)u^8(-du)$$

$$= -\int (1-u)u^8 du$$

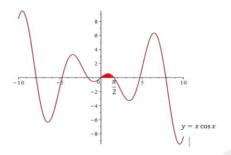
$$= -\int u^8 - u^9 du$$

$$= -\left[\frac{u^9}{9} - \frac{u^{10}}{10}\right] + C$$

$$= -\left[\frac{(1-x)^9}{9} - \frac{(1-x)^{10}}{10}\right] + C$$

$$= \frac{(1-x)^{10}}{10} - \frac{(1-x)^9}{9} + C$$

(b)



$$Area = \int_0^{\frac{\pi}{2}} x \cos x \ dx$$

$$\int x \cos x \ dx$$

$$u = x$$
 $dv = \cos x \, dx$
 $\frac{du}{dx} = 1$ $\int dv = \int \cos x \, dx$
 $du = dx$ $v = \sin x$

$$\int u \ dv = uv - \int v \ du$$

$$Area = (x)(\sin x) - \int (\sin x) (dx)$$
$$= x \sin x - (-\cos x)$$
$$= x \sin x + \cos x$$

$$Area = \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= [x \sin x + \cos x]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right] - [0 \sin 0 + \cos(0)]$$

$$= \left[\frac{\pi}{2} + 0\right] - [0 + 1]$$

$$= \frac{\pi}{2} - 1 \ unit^2$$

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2. Solve $\frac{dy}{dx} + 2(x+1)y^2 = 0$, given that y = 1 when x = 1. Express y in terms of x.

SOLUTION

$$\frac{dy}{dx} + 2(x+1)y^2 = 0$$

$$\frac{dy}{dx} = -2(x+1)y^2$$

$$y^{-2}dy = -2(x+1)dx$$

$$\int y^{-2}dy = -2\int (x+1)dx$$

$$\frac{y^{-1}}{-1} = -2\left(\frac{x^2}{2} + x\right) + C$$

$$-\frac{1}{y} = -x^2 - 2x + C$$

When
$$x = 1, y = 1$$

$$-\frac{1}{1} = -1^{2} - 2(1) + C$$

$$-1 = -1 - 2 + C$$

$$C = 2$$

$$-\frac{1}{y} = -x^{2} - 2x + 2$$

$$-1 = (-x^{2} - 2x + 2)y$$

3. Use Newton-Raphson method to solve the equation $e^x - x - 2 = 0$ correct to four decimal places by taking $x_1 = 1$ as the first approximation.

SOLUTION

$$e^{x} - x - 2 = 0$$

$$f(x) = e^{x} - x - 2$$

$$f'(x) = e^{x} - 1$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = x_n - \frac{e^{x} - x - 2}{e^{x} - 1}$$

$$x_{1} = 1$$

$$x_{2} = 1 - \frac{e^{1} - 1 - 2}{e^{1} - 1} = 1.16395$$

$$x_{3} = 1.16395 - \frac{e^{1.16395} - 1.16395 - 2}{e^{1.163951} - 1} = 1.14642$$

$$x_{4} = 1.14642 - \frac{e^{1.14642} - 1.14642 - 2}{e^{1.14642} - 1} = 1.14619$$

$$x_{4} = 1.14619 - \frac{e^{1.14619} - 1.14619 - 2}{e^{1.14619} - 1} = 1.14619$$

$$x = 1.1462$$

- 4. An ellipse $Ax^2 + y^2 + Bx + Cy + 1 = 0$ passes through points (0,1), (1,-1) and (2,1).
 - a) Find the equation of the ellipse in the standard form. Hence, state the centre and vertices of the ellipse.
 - b) Find the foci of th ellipse.
 - c) Sketch the graph of the ellipse.

$$Ax^{2} + y^{2} + Bx + Cy + 1 = 0$$

$$At (0,1)$$

$$A(0^{2}) + 1^{2} + B0 + C(1) + 1 = 0$$

$$C = -2$$

$$At (1,-1) and C = -2$$

$$A(1^{2}) + (-1)^{2} + B(1) + (-2)(-1) + 1 = 0$$

$$A + 1 + B + 2 + 1 = 0$$

$$A + B = -4$$
.....(1)

At (2,1) and
$$C = -2$$

$$A(2)^{2} + (1)^{2} + B(2) + (-2)(1) + 1 = 0$$

$$4A + 1 + 2B - 2 + 1 = 0$$

$$4A + 2B = 0$$
(2)

(1)X2

$$2A + 2B = -8$$
(3)

$$(2) - (3)$$
$$2A = 8$$
$$A = 4$$
$$B = -8$$

The equation of ellipse:

$$4x^{2} + y^{2} - 8x - 2y + 1 = 0$$

$$4x^{2} - 8x + y^{2} - 2y = -1$$

$$4(x^{2} - 2x) + (y^{2} - 2y) = -1$$

$$4(x^{2} - 2x + 1^{2} - 1^{2}) + (y^{2} - 2y + 1^{2} - 1^{2}) = -1$$

$$4[(x-1)^2-1]+[(y-1)^2-1]=-1$$

$$4(x-1)^2 - 4 + (y-1)^2 - 1 = -1$$

$$4(x-1)^2 + (y-1)^2 = 4$$

$$\frac{4(x-1)^2}{4} + \frac{(y-1)^2}{4} = \frac{4}{4}$$

$$\frac{(x-1)^2}{1^2} + \frac{(y-1)^2}{2^2} = 1$$

Standard Equation of Ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$a = 1, b = 2, h = 1, k = 1$$

Centre:
$$C(h, k) = C(1,1)$$

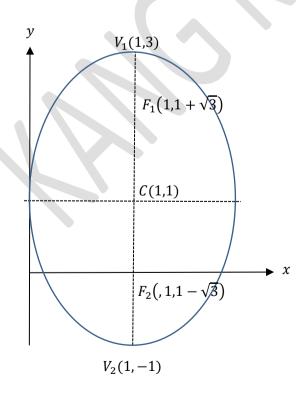
Vertices:
$$V(h, k \pm b) \Rightarrow V_1(1,1+2), V_2(1,1-2)$$

 $V_1(1,3), V_2(1,-1)$

$$c = \sqrt{2^2 - 1^2} = \sqrt{3}$$

Foci:
$$F(h, k \pm c) = F(1, 1 \pm \sqrt{3})$$

(4c)



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- 5. The line L_1 and L_2 passes through the point R(2, 4, -3) and S(8, -5, 9) in the direction of $2\mathbf{i} 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{i} 2\mathbf{j} + 3\mathbf{k}$, respectively.
 - a) State the equations for lines L_1 and L_2 in the vector form. Hence, calculate the acute angle between the lines L_1 and L_2 .
 - b) Find the equation of plane containing the line L_1 and the point (7, -3, 5) in the Cartesian form.
 - c) Determine whether the line L_2 is parallel to the plane x + 5y + 3z = 5.

(5a)

$$L_1$$
: $a_1 = 2i + 4j - 3k$; $v_1 = 2i - 3j + 4k$
 L_2 : $a_2 = 8i - 5j + 9k$; $v_2 = i - 2j + 3k$
 $r = a + tv$

Equation of L_1 and L_2

$$L_1$$
: $r_1 = (2i + 4j - 3k) + t_1(2i - 3j + 4k)$
 L_2 : $r_2 = (8i - 5j + 9k) + t_2(i - 2j + 3k)$

Angle between L_1 and $L_2(\theta)$

$$Cos \theta = \frac{v_1 \cdot v_2}{|v_1||v_2|}$$

$$Cos \theta = \frac{(2i - 3j + 4k) \cdot (i - 2j + 3k)}{|2i - 3j + 4k||i - 2j + 3k|}$$

$$= \frac{(2)(1) + (-3)(-2) + (4)(3)}{\sqrt{(2)^2 + (-3)^2 + (4)^2}\sqrt{(1)^2 + (-2)^2 + (3)^2}}$$

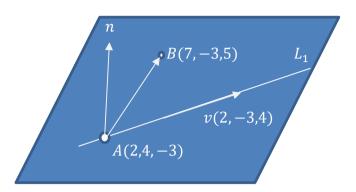
$$= \frac{20}{\sqrt{29}\sqrt{14}}$$

$$= 0.9926$$

$$\theta = cos^{-1}(0.9926)$$

$$= 6.98^{\circ}$$

(5b)



Equation of plane

r.n = a.n

$$a = 7i - 3j + 5k$$

$$n = v \times \overrightarrow{AB}$$

$$v = 2i - 3j + 4k$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (7i - 3j + 5k) - (2i + 4j - 3k)$$

$$= (5i - 7j + 8k)$$

$$n = \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 5 & -7 & 8 \end{vmatrix}$$

$$= (-24 + 28)i - (16 - 20)j + (-14 + 15)k$$

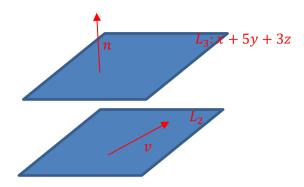
$$= 4i + 4j + k$$

Equation of plane

$$r. n = a. n$$

 $(xi + yj + zk). (4i + 4j + k) = (7i - 3j + 5k). (4i + 4j + k)$
 $4x + 4y + z = 28 - 12 + 5$
 $4x + 4y + z = 21$

(5c)



From the diagram, if L_2 is parallel to L_3 , then v must be perpendicular to n

$$\boldsymbol{v} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$n = i + 5j + 3k$$

$$v.n = (i - 2j + 3k).(i + 5j + 3k)$$

$$= 1 - 10 + 9$$

$$= 0$$

Since v.n = 0, there for line L_1 is parallel to the plane x + 5y + 3z = 5.

6. **TABLE 1** shows the frequency distribution of the diameter of 100 pebbles which are measured to the nearest milliType equation here meter (mm).

TABLE 1

Diameter (mm)	Frequency
10 – 14	22
15 – 19	20
20 – 24	25
25 – 29	15
30 - 34	18

Calculate the

- a) mean.
- b) mode.
- c) median.

SOLUTION

Diameter	Class Boundary	Mid Point	Frequency	Cummulative
(mm)		(x)	(f)	Frequency
				(F)
10 – 14	9.5 – 14.5	12	22	22
15 – 19	14.5 – 19.5	17	20	42
20 – 24	19.5 – 24.5	22	25	67
25 – 29	24.5 – 29.5	27	15	82
30 - 34	30.5 – 34.5	32	18	100
		$\sum f$	100	

(6a)

$$Mean = \frac{\sum fx}{\sum f}$$

$$Mean = \frac{(12)(22) + (17)(20) + (22)(25) + (27)(15) + (32)(18)}{100}$$
$$= \frac{2135}{100}$$
$$= 21.35$$

(6b)

Mode =
$$L_m + \left[\frac{d_1}{d_1 + d_2} \right] C$$

Mode = $19.5 + \left[\frac{5}{5 + 10} \right] 5$
= 21.17

(6c)

$$Median = L_m + \left[\frac{\frac{n}{2} - F_{k-1}}{f_k}\right]C$$

$$Median = 19.5 + \left[\frac{\frac{100}{2} - 42}{25}\right]5$$

$$= 21.1$$

- 7. A committee of 5 is to be selected from a group of 7 men and 6 women. How many different committees could be formed if
 - a) there is no woman in the committee?
 - b) a particular man must be in the committee and the remaining has equal number of men and women?
 - c) at least 3 men are in the committee?

(7a)

if there is no woman in the committee.

$$^{7}C_{5}$$
 $^{6}C_{0}$ =21

(7b)

$${}^{1}C_{1} x {}^{6}C_{2} x {}^{6}C_{2} = 225$$

(7c)

At least 3 men in the committee.

Men	Women	Total
3	2	$^{7}C_{3}$ $^{6}C_{2} = 525$
4	1	$^{7}C_{4}$ $^{6}C_{1}$ =210
5	0	$^{7}C_{5}$ $^{6}C_{0}$ =21
	Total	756

The number of different committee could be formed = 756

8. The probabilities of events X and Y are given as $P(X) = \frac{3}{5}$, $P(X'|Y) = \frac{31}{45}$ and

$$P(X \cap Y) = \frac{2}{25}.$$

- a) Show that $P(Y) = \frac{9}{35}$
- b) Find $P(X \cup Y')$.

SOLUTION

(8a)

$$P(X) = \frac{3}{5}$$

$$P(X \cap Y) = \frac{2}{25}$$

$$P(X'|Y) = \frac{31}{45}$$

$$\frac{P(X' \cap Y)}{P(Y)} = \frac{31}{45}$$

$$\frac{P(Y) - P(X \cap Y)}{P(Y)} = \frac{31}{45}$$

$$\frac{P(Y) - \frac{2}{25}}{P(Y)} = \frac{31}{45}$$

$$1 - \frac{\frac{2}{25}}{P(Y)} = \frac{31}{45}$$

$$\frac{\frac{2}{25}}{P(Y)} = 1 - \frac{31}{45}$$

$$\frac{\frac{2}{25}}{P(Y)} = \frac{14}{45}$$

$$P(Y) = \frac{2}{25}x\frac{45}{14}$$

$$P(Y) = \frac{9}{35}$$

(8b)

$$P(X \cup Y') = P(X) + P(Y') - P(X \cap Y')$$

$$P(Y') = 1 - P(Y) = 1 - \frac{9}{35} = \frac{26}{35}$$

$$P(X \cap Y') = P(X) - P(X \cap Y) = \frac{3}{5} - \frac{2}{25} = \frac{13}{25}$$

$$P(X \cup Y') = \frac{3}{5} + \frac{26}{35} - \frac{13}{25}$$
$$= \frac{144}{175}$$



9. A discrete random variable X has the probability distribution function

$$f(x) = \begin{cases} \frac{x+1}{16}, & x = 2, 3, 4\\ kx, & x = 6, 8\\ 0, & otherwise \end{cases}$$

- a) Show that $k = \frac{1}{56}$
- b) Hence, calculate $P(3 \le X < 8)$.
- c) Determine the values of E(X) and Var(X). Thus, evaluate $Var(\sqrt{3}X 1)$

SOLUTION

$$f(x) = \begin{cases} \frac{x+1}{16}, & x = 2, 3, 4\\ kx, & x = 6, 8\\ 0, & otherwise \end{cases}$$

x	2	3	4	6	8
P(X=x)	3 16	$\frac{4}{16}$	5 16	6 <i>k</i>	8 <i>k</i>

(9a)

$$\sum P(X=x)=1$$

$$\frac{3}{16} + \frac{4}{16} + \frac{5}{16} + 6k + 8k = 1$$

$$\frac{12}{16} + 14k = 1$$

$$14k = \frac{1}{4}$$

$$k = \frac{1}{56}$$

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(9b)

Ī	x	2	3	4	6	8
	P(X=x)	$\frac{3}{16}$	$\frac{4}{16}$	5 16	$\frac{3}{28}$	$\frac{1}{7}$

$$P(3 \le X < 8) = P(X = 3) + P(X = 4) + P(X = 6)$$
$$= \frac{4}{16} + \frac{5}{16} + \frac{3}{28}$$
$$= \frac{75}{112}$$

(9c)

$$E(X) = \sum x P(X = x)$$

$$E(X) = (2)\left(\frac{3}{16}\right) + (3)\left(\frac{4}{16}\right) + (4)\left(\frac{5}{16}\right) + (6)\left(\frac{3}{28}\right) + (8)\left(\frac{1}{7}\right)$$
$$= \frac{233}{56}$$

$$E(X^2) = \sum x^2 P(X = x)$$

$$E(X^{2}) = (2)^{2} \left(\frac{3}{16}\right) + (3)^{2} \left(\frac{4}{16}\right) + (4)^{2} \left(\frac{5}{16}\right) + (6)^{2} \left(\frac{3}{28}\right) + (8)^{2} \left(\frac{1}{7}\right)$$
$$= 21$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 21 - \left(\frac{233}{56}\right)^2$$
$$= 3.688$$

$$Var(ax + b) = a^2 Var(x)$$

$$Var(\sqrt{3}x - 1) = \sqrt{3}^{2}x \ 3.688$$
$$= 11.065$$

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- 10. The number of batteries sold at a service center on any particular day follows a Poisson distribution with mean λ .
 - a) If the probability of selling exactly 4 batteries divided by the probability of selling exactly 2 batteries is $\frac{225}{12}$, show that $\lambda=15$.
 - b) On any particular day, calculate the probability that the service center sells between 5 and 14 batteries.
 - c) Given that the probability of selling less than k batteries on any particular day is 0.917, find the value of k.
 - d) Find the probability that exactly 40 batteries are sold in 2 working days. Give your answer in four decimal places.

(10a)

For Poisson Distribution

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\frac{P(X=4)}{P(X=2)} = \frac{225}{12}$$

$$\frac{\frac{e^{-\lambda}\lambda^4}{4!}}{\frac{e^{-\lambda}\lambda^2}{2!}} = \frac{225}{12}$$

$$\left(\frac{e^{-\lambda}\lambda^4}{4!}\right)\left(\frac{2!}{e^{-\lambda}\lambda^2}\right) = \frac{225}{12}$$

$$\frac{\lambda^2}{12} = \frac{225}{12}$$

$$\lambda^2 = 225$$

$$\lambda = 15$$

(10b)

$$X \sim Po(15)$$

 $P(5 < X < 14) = P(X \ge 6) - P(X \ge 14)$
 $= 0.9975 - 0.6368$
 $= 0.3604$

(10c)

$$P(X < k) = 0.917$$

$$1 - P(X \ge k) = 0.917$$

$$P(X \ge k) = 0.0830$$

From statistical table:

$$k = 21$$

(10d)

$$X \sim Po(30)$$

$$P(X = 40) = \frac{e^{-30}(30)^{40}}{40!}$$
$$= 0.0139$$