Chapter 5: Functions and Graphs

5.4 Exponential and Logarithmic Functions

Prepared by: Kang Kooi Wei

Learning Outcomes

(a) Find the relationship of exponential and logarithmic functions by algebraic and graphical approaches.

*Highlight the fact that one is the inverse of the other function.

**To discuss functions of the form $e^{g(x)}$ and $ln(g(x))$ where $g(x) = ax + b$.

(b) State the domain and range of an exponential and logarithmic functions.

(c) Compute the composite functions involving exponential and logarithmic functions.

(d) Sketch the graph which involve exponential and logarithmic functions on the same axes.

*Such as: $y = a^{mx+c}, y = e^{mx+c}$ and $y = ln(mx + c)$.

Bloom: Understanding

Exponential Functions

An exponential function has a variable in an exponent.

Exponential

$$f(x) = 2^x$$

Base

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Some examples of exponential functions are

$$f(x) = 3^x, f(x) = 5^{2x}, f(x) = \left(\frac{1}{2}\right)^x, f(x) = 2^{3-x}$$

Bloom: Remembering

Bloom: Understanding

Domain and Range of Exponential Functions

Domain: $(-\infty, \infty)$

To determine Range:

Example:

1. $f(x) = e^x$

Let $e^x > 0$

$f(x) > 0$

$\therefore Range: (0, \infty)$

2. $f(x) = e^x - 5$

Let $e^x > 0$

$e^x - 5 > -5$

$f(x) > -5$

$\therefore Range: (-5, \infty)$

Bloom: Remembering

Bloom: Understanding

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Logarithmic Functions

A logarithmic function is a function of the form

$$f(x) = log_a x \quad \text{where} \quad a > 0 \; and \; a \neq 1$$

- Constant a is known as the base
- Variable x is any positive real numbers.

Some examples of logarithmic functions are

$$f(x) = log_2 x, f(x) = log_5(x-3)$$

Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

Bloom: Remembering
Bloom: Understanding

Domain and Range of Logarithmic Functions

Range: $(-\infty, \infty)$

To determine Domain:

Example:

1. $f(x) = log_2 x$

   Let $x > 0$

   $\therefore Domain: (0, \infty)$

2. $f(x) = log_5(x-3)$

   Let $x - 3 > 0$

   $x > 3$

   $\therefore Domain: (3, \infty)$

Bloom: Remembering

Bloom: Understanding

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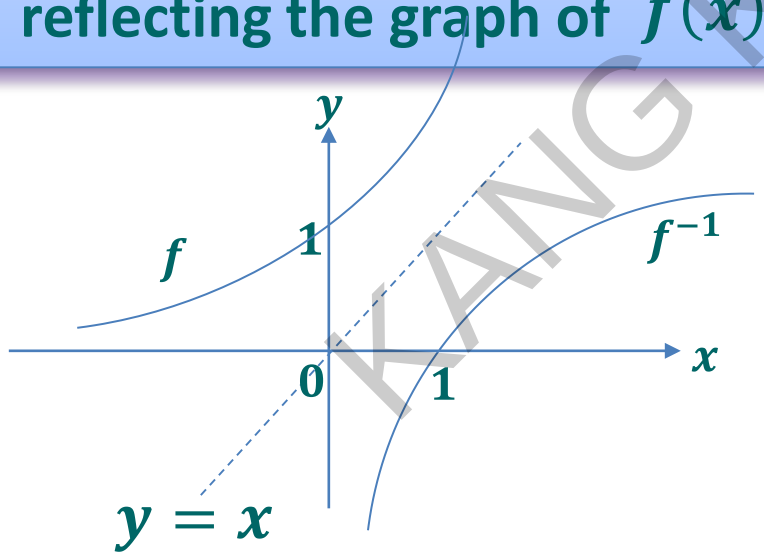
Relationship between an exponential function and a logarithmic function

- **Exponential function** $f(x) = a^x$ **is a one-to-one function, it has an inverse function.**
- **The inverse function of** $f(x) = a^x$ **is a logarithmic function** $f^{-1}(x) = log_a x$ **.**
- **Therefore, the graph of** $f^{-1}(x) = log_a x$ **is obtained by reflecting the graph of** $f(x) = a^x$ **in the line** $y = x$ **.**



Bloom: Remembering
Bloom: Understanding

Example

1. Sketch the graphs of the following functions:

   (a) $y = e^x$

   (b) $y = e^x + 3$

   (c) $y = e^x - 3$

2. Sketch the following graphs:

   (a) $y = lnx$

   (b) $y = ln(x + 3)$

   (c) $y = ln(x - 2)$

Bloom: Understanding

Solution

1(a). Steps for sketching the graph of $y = e^x$:

Step 1: Find the domain:

$$D_f = (-\infty, \infty)$$

Step 2: Find the $x - intercept$ and $y - intercept$:

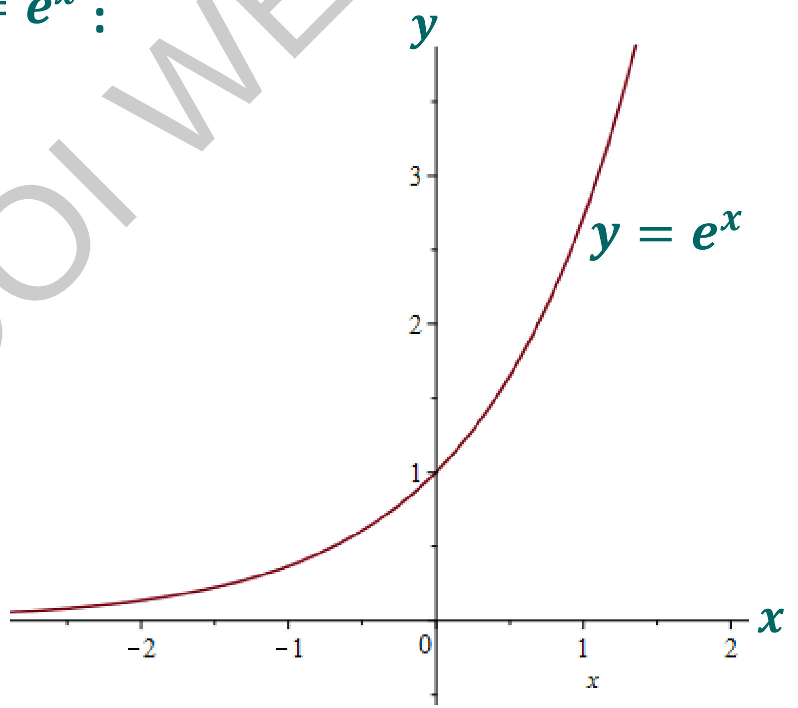
When $x = 0, y = 1$

When $y = 0$, x is undefined.

Step 3: When $x \to -\infty, y \to 0$

Implies that $y = 0$ is the horizontal asymptote.

When $x \to +\infty, y \to +\infty$



Bloom: Understanding

Solution

1(b). **Steps for sketching the graph of $y = e^x + 3$:**

Step 1: Find the domain:

$$D_f = (-\infty, \infty)$$

Step 2: Find the $x - intercept$ and $y - intercept$:

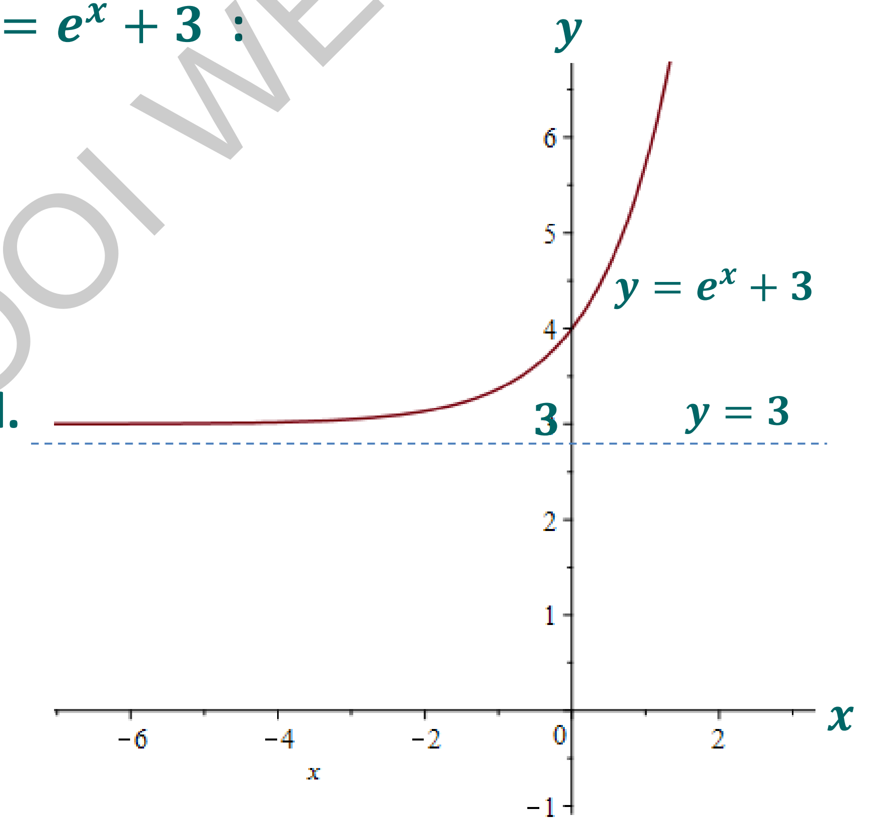
When $x = 0, y = 1 + 3 = 4$

When $y = 0$, x is undefined.

Step 3: When $x \to -\infty, y \to 3$

Implies that $y = 3$ is the horizontal asymptote.

When $x \to +\infty, y \to +\infty$



Bloom: Understanding

Solution

1(c). Steps for sketching the graph of $y = e^x - 3$:

Step 1: Find the domain:

$$D_f = (-\infty, \infty)$$

Step 2: Find the $x - intercept$ and $y - intercept$:

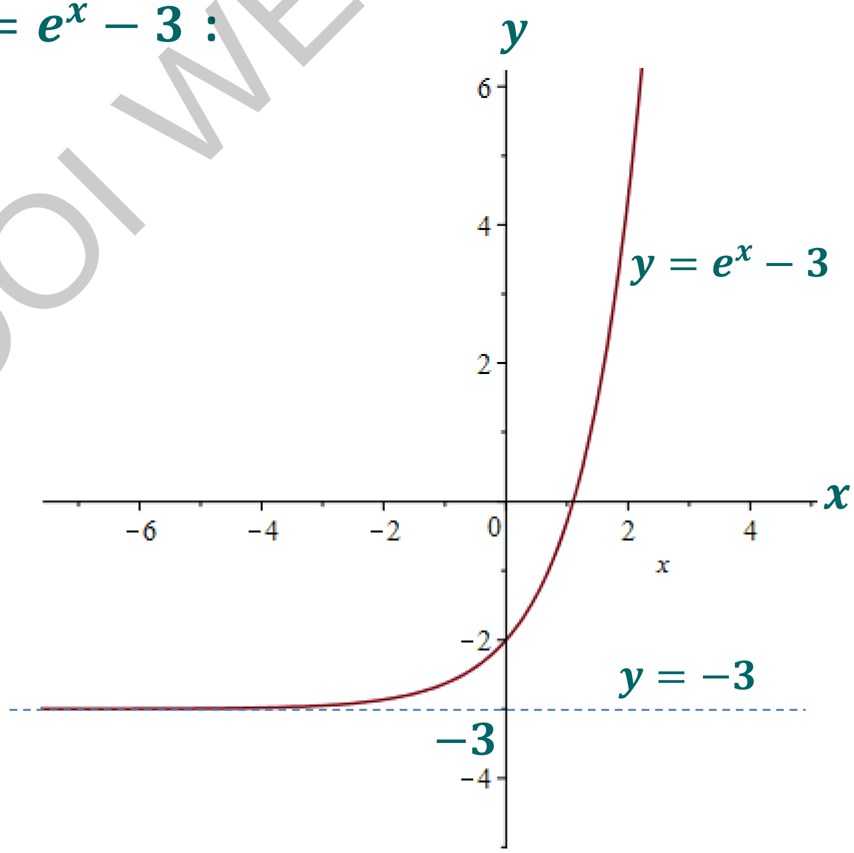
When $x = 0, y = 1 - 3 = -2$

When $y = 0$, x is undefined.

Step 3: When $x \to -\infty, y \to -3$

Implies that $y = -3$ is the horizontal asymptote.

When $x \to +\infty, y \to +\infty$



Bloom: Understanding

Solution

2(a). Steps for sketching the graph of $y = lnx$:

Step 1: Find the domain:

$$D_f = (0, \infty)$$

Step 2: Find the $x - intercept$ and $y - intercept$:

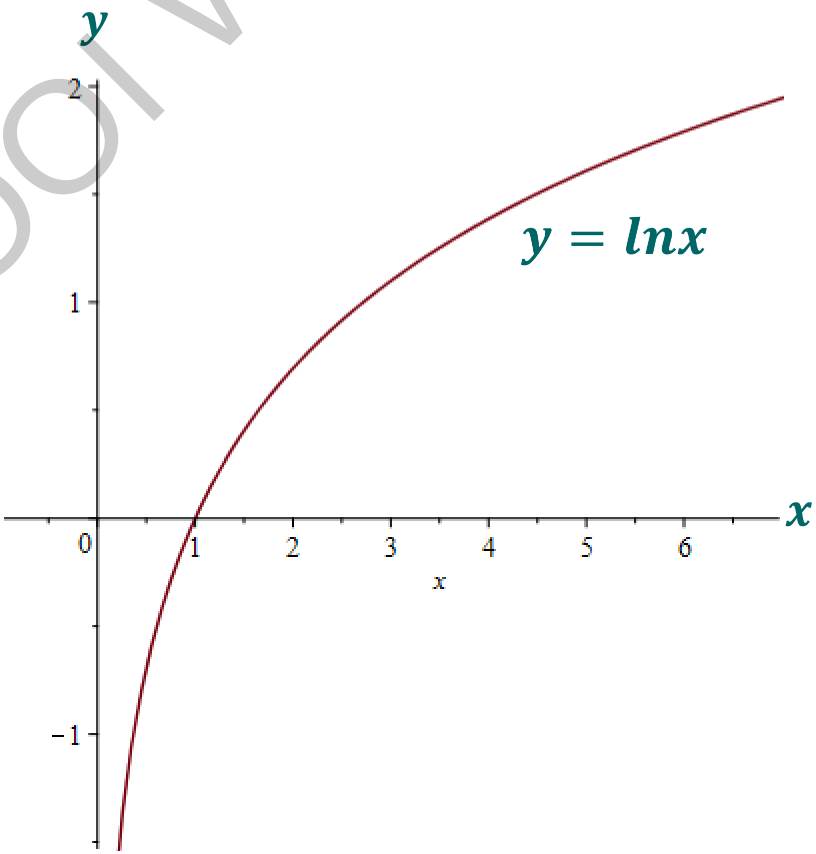
When $x = 0, y$ is undefined.

When $y = 0$, $x = 1$

Step 3: When $x \to 0, y \to -\infty$

Implies that $x = 0$ is the vertical asymptote.

When $x \to +\infty, y \to +\infty$



Bloom: Understanding

Solution

2(b). Steps for sketching the graph of $y = ln(x+3)$:

Step 1: Find the domain:

$$D_f = (-3, \infty)$$

Step 2: Find the $x - intercept$ and $y - intercept$:

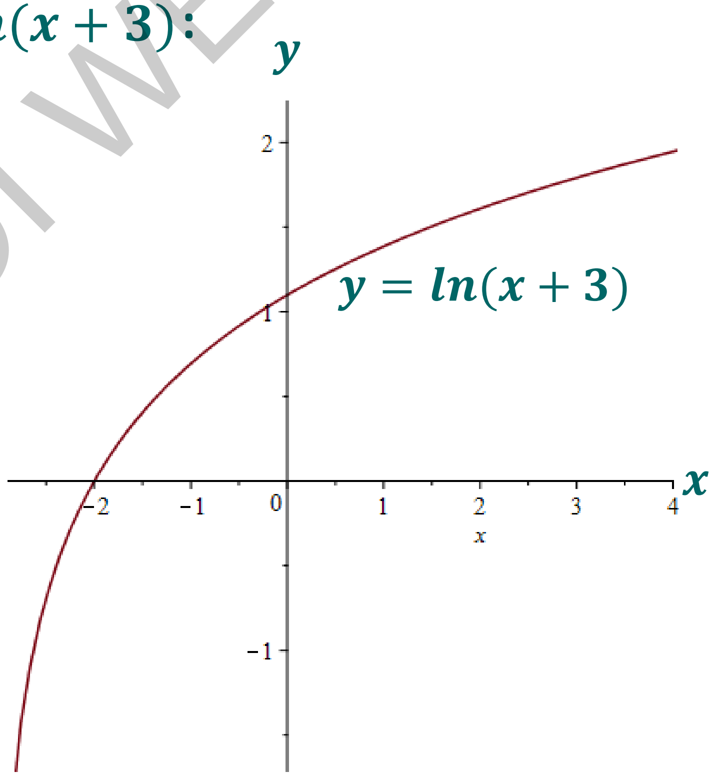
When $x = 0, y = ln3$

When $y = 0$, $x = -2$

Step 3: When $x \to -3, y \to -\infty$

Implies that $x = -3$ is the vertical asymptote.

When $x \to +\infty, y \to +\infty$



Bloom: Understanding

Solution

2(c). Steps for sketching the graph of $y = ln(x-2)$:

Step 1: Find the domain:

$$D_f = (2, \infty)$$

Step 2: Find the $x - intercept$ and $y - intercept$:

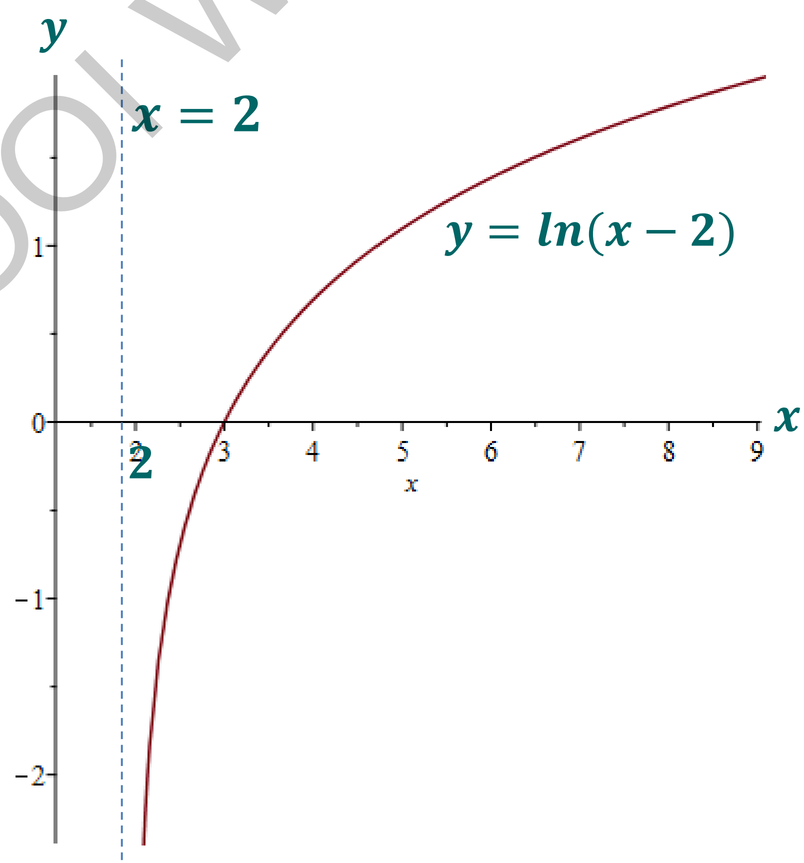
When $x = 0, y$ is undefined.

When $y = 0$, $x = 3$

Step 3: When $x \to 2, y \to -\infty$

Implies that $x = 2$ is the vertical asymptote.

When $x \to +\infty, y \to +\infty$



Bloom: Understanding

Example

1. The function f is defined as $f(x) = 2lnx - 1$.
   (a) State the domain for f.
   (b) State the range for f.
   (c) Show that f^{-1} exists.
   (d) Find f^{-1}.
   (e) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes.

Bloom: Understanding

Solution

1. (a) $D_f = (0, \infty)$

For $x > 0$.

(b) $R_f = (-\infty, \infty)$

(c) Let $x_1, x_2 \in D_f$ and when $f(x_1) = f(x_2)$

$$2ln(x_1) - 1 = 2ln(x_2) - 1$$

$$ln(x_1) = ln(x_2)$$

$$x_1 = x_2$$

Hence f is a one-to-one function and f^{-1} exists.

Bloom: Understanding

Solution (Continue…)

1. (d) $f\left(f^{-1}(x)\right) = x$

$$2ln\left(f^{-1}(x)\right) - 1 = x$$

$$ln\left(f^{-1}(x)\right) = \frac{x+1}{2}$$

$$f^{-1}(x) = e^{\frac{x+1}{2}}$$

Bloom: Understanding

Solution (Continue...)

1. (e) Steps for sketching the graph of $f(x) = 2lnx - 1$:

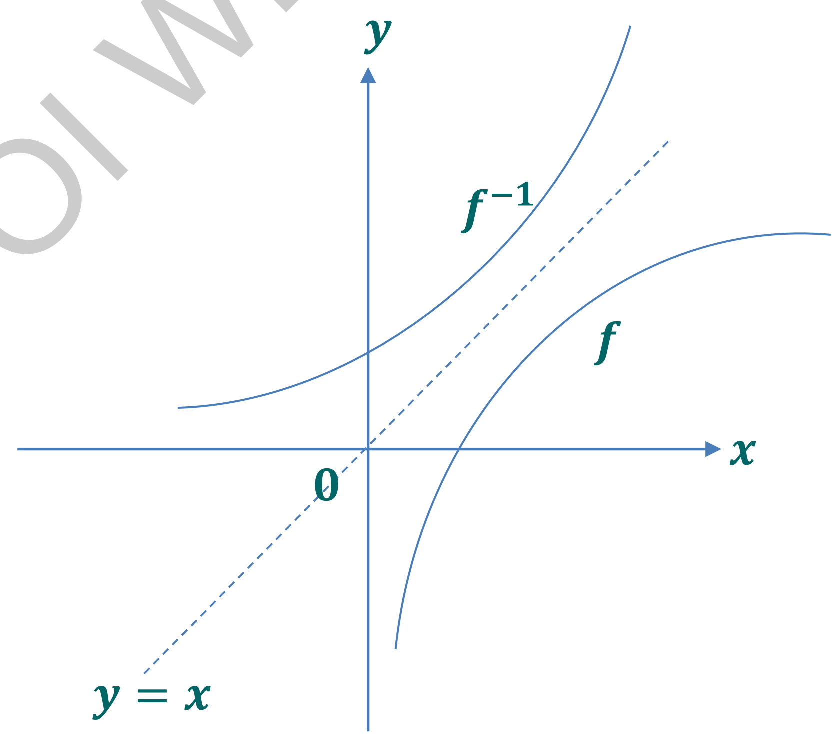
Step 1: Find the domain:

$$D_f = (0, \infty)$$

Step 2: Find the $x - intercept$:

When $y = 0, \; x = e^{\frac{1}{2}} = 1.65$

Step 3: When $x \to 0, y \to -\infty$
Implies that $x = 0$ is the vertical asymptote.
When $x \to +\infty, y \to +\infty$



The graph of $y = f^{-1}(x)$ is a reflection of the graph $y = f(x)$ in the line $y = x$.

Bloom: Understanding

Self-check

1. Sketch the graphs of the following functions:

(a) $y = e^x$

(b) $y = e^{x+3}$

(c) $y = e^{x-3}$

2. Sketch the following graphs:

(a) $y = lnx$

(b) $y = lnx + 3$

(c) $y = lnx - 2$

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Bloom: Applying

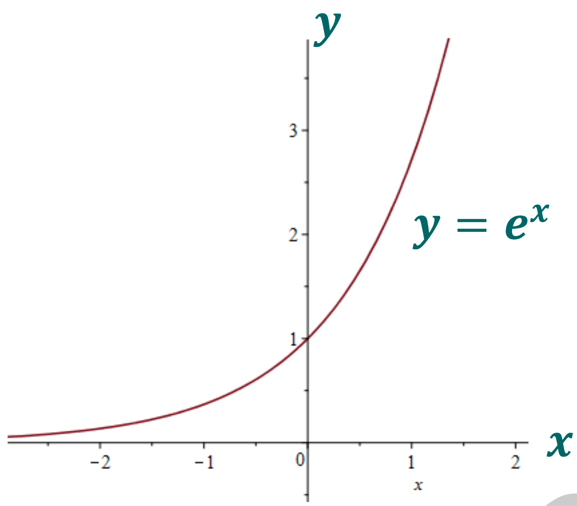
Self-check

3. The function f is defined as $f(x) = 2e^{x} - 1$.
   (a) State the domain for f.
   (b) State the range for f.
   (c) Show that f^{-1} exists.
   (d) Find f^{-1}.
   (e) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes.

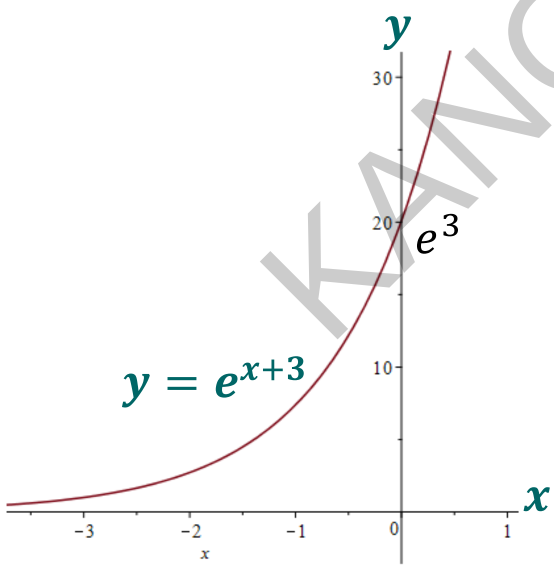
Bloom: Applying

Answer Self-check

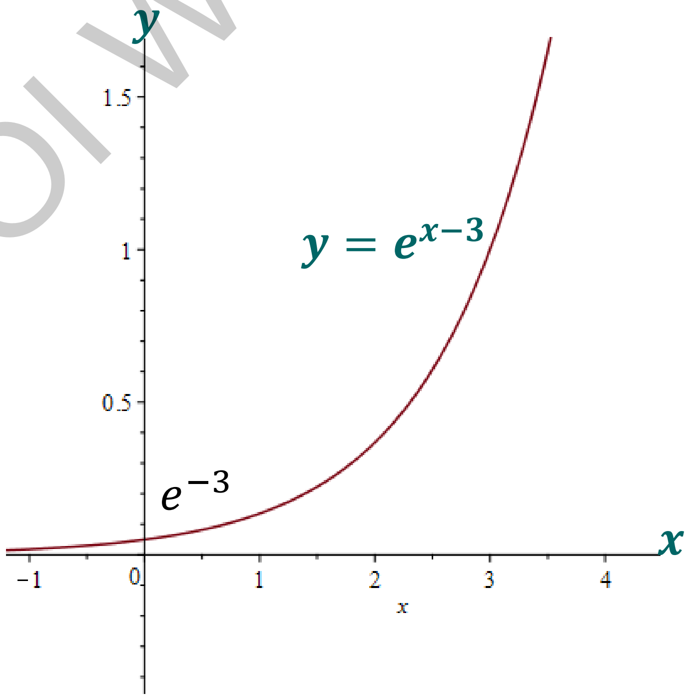
1(a)



1(b)



1(c)



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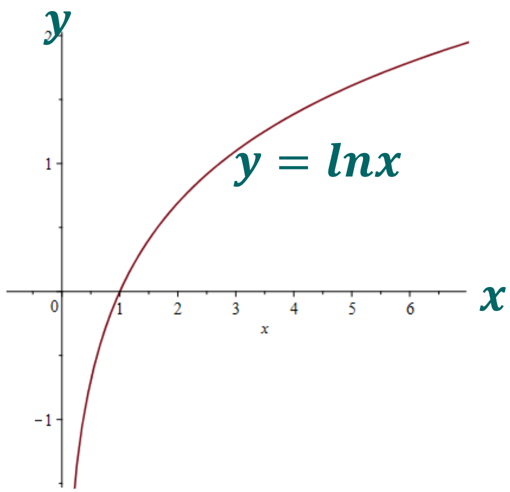
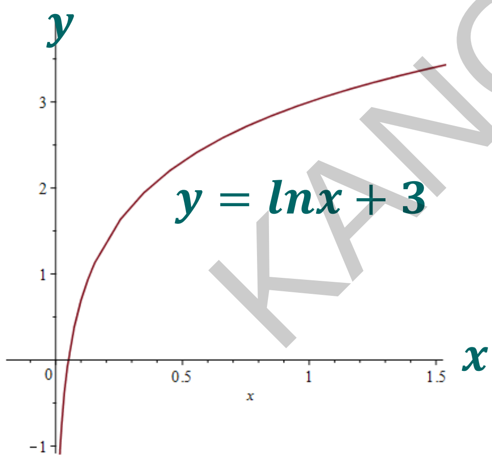
Bloom: Applying

Answer Self-check

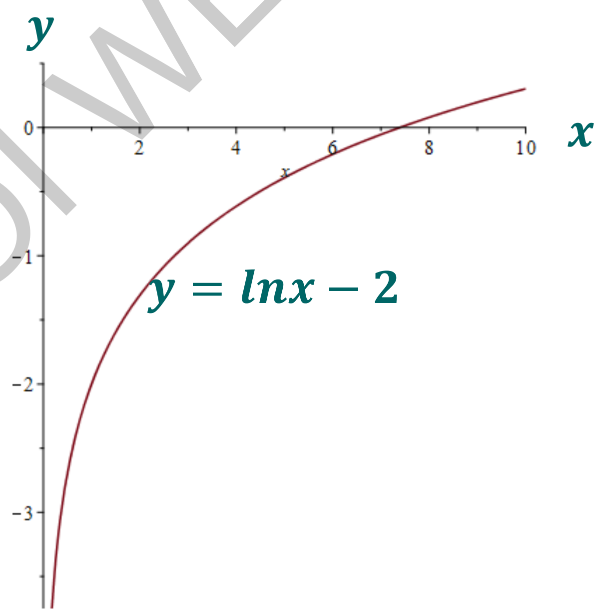
2(a)



2(b)



2(c)



Bloom: Applying

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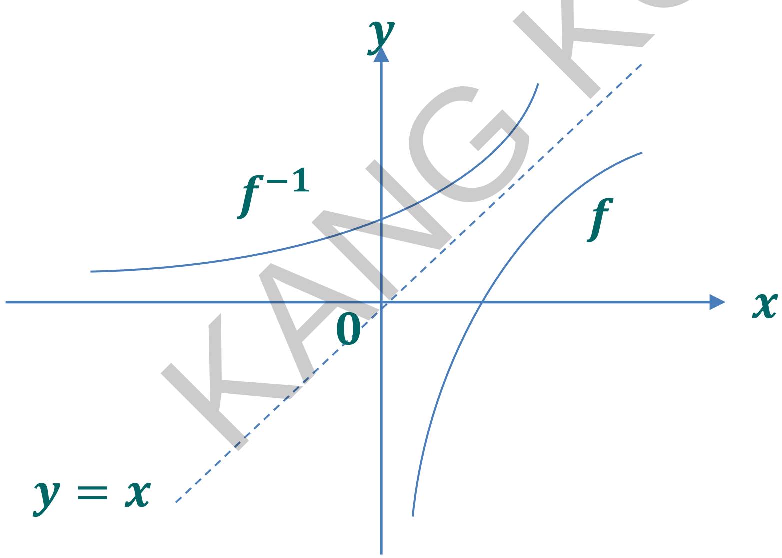
Answer Self-check

3. (a) $D_f = (-\infty, \infty)$

(b) $R_f = (-1, \infty)$

(d) $f^{-1}(x) = ln\left(\frac{x+1}{2}\right)$

(e)



Bloom: Applying

Summary

Domain of exponential function always same: $(-\infty, \infty)$

Range of logarithmic function always same: $(-\infty, \infty)$

Exponential and Logarithmic Functions

Inverse of exponential function is logarithmic function and vice versa.

The graph of exponential function is a reflection of the graph of its inverse (logarithmic function) in the line $y = x$.

Key Terms

- Exponential Functions
- Logarithmic Functions
- Inverse Functions
- Domain
- Range
- Composite Functions
- Sketching graph