# Chapter 5: Functions and Graphs

## 5.1 Functions



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# Learning Outcomes

(a) Define a function.

** Emphasize the concept of one-to-one and onto.*

(b) Identify a function from the graph by using vertical line test.

(c) Identify a one-to-one function by using algebraic approach or horizontal line test.

(d) Sketch the graph of a function.

** Include polynomials up to degree 3, functions such as piecewise, absolute values of linear functions, reciprocal, square root functions with linear expression.*

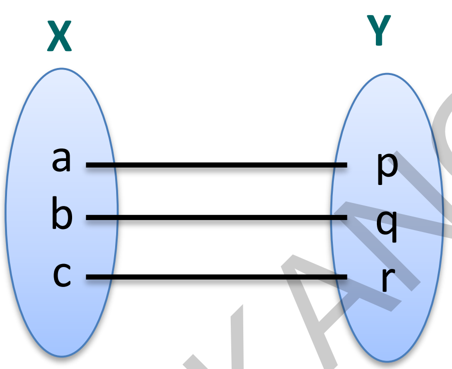
(e) State the domain and range of a function.

** Use algebraic or graphical approach to find domain and range.*

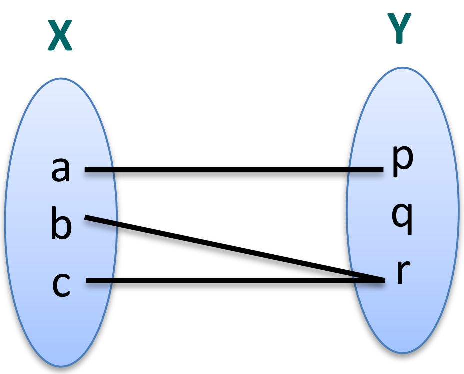
Bloom: Understanding

# Functions

**A function is a relation where each element of one set is mapped to exactly one and only one element of another set. Consequently, relations which are one-to-one and many-to-one are functions.**




**One-to-one onto function**




**Many-to-one function**



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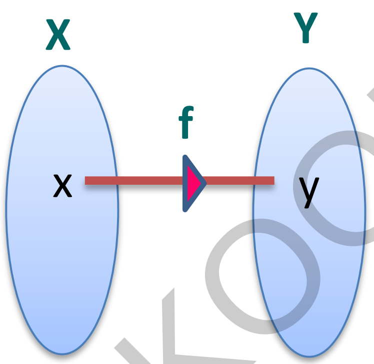
# Function notation




The mapping of x to y under the function f can be written as
$f: x \rightarrow y$ where $y = f(x)$

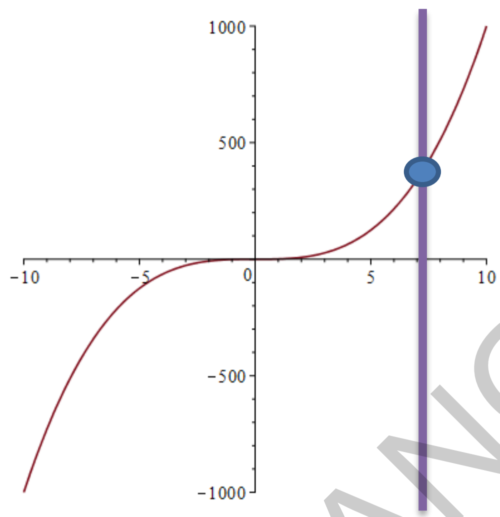
The set of all elements of X is called the domain of f.

The set of all elements of Y is called the codomain of f.

The set of all the images y is called the range of f.

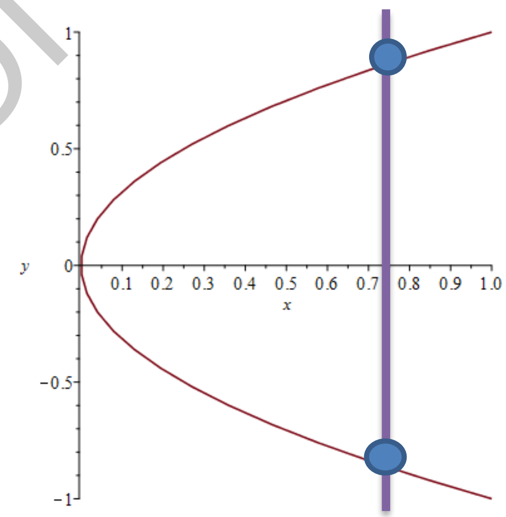
# Vertical line test for a function

**Any vertical line $x = a$ intersects the graph of a function at most one point.**




**A vertical line intersects the graph at one point.**

**∴ the graph represents a function.**




**A vertical line intersects the graph at two point.**

**∴the graph does not represent a function.**

# One-to-one function test

**Algebraic Method** (Method 1)

A function f is one-to-one if

$$f(x_1) = f(x_2) \text{ Implies that } x_1 = x_2$$

**Horizontal line test** (Method 2)

Any horizontal line $y = b$ intersects the graph of a function at most one point, then f is one-to-one function.

# Example

(1) By using the algebraic method, determine whether $f$ is a one-to-one function or not.

(a) $f(x) = \frac{x}{2x+1}$

(b) $f(x) = |x - 3|$

(2) Use graphical method to determine whether each of the following functions is a one-to-one function.

(a) $f(x) = x(x+4)$

(b) $f(x) = x^3 + 1$

# Solution

(1)(a)

$$f(x_1) = f(x_2)$$

$$\frac{x_1}{2x_1+1} = \frac{x_2}{2x_2+1}$$

$$x_1(2x_2+1) = x_2(2x_1+1)$$

$$2x_1x_2 + x_1 = 2x_1x_2 + x_2$$

$$x_1 = x_2$$

Let $f(x_1) = f(x_2)$.

Cross multiplying.

Expanding both sides.

**Since $f(x_1) = f(x_2)$ implies $x_1 = x_2$, $f$ is a one-to-one function.**





*Bloom: Understanding*

# Solution

(1)(b) $f(x_1) = f(x_2)$

Let $f(x_1) = f(x_2)$.

$|x_1 - 3| = |x_2 - 3|$

$x_1 - 3 = x_2 - 3$ or $x_1 - 3 = -(x_2 - 3)$

$x_1 = x_2$ $\quad$ $x_1 = -x_2 + 6$

**Since $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$ , $f$ is not a one-to-one function.**





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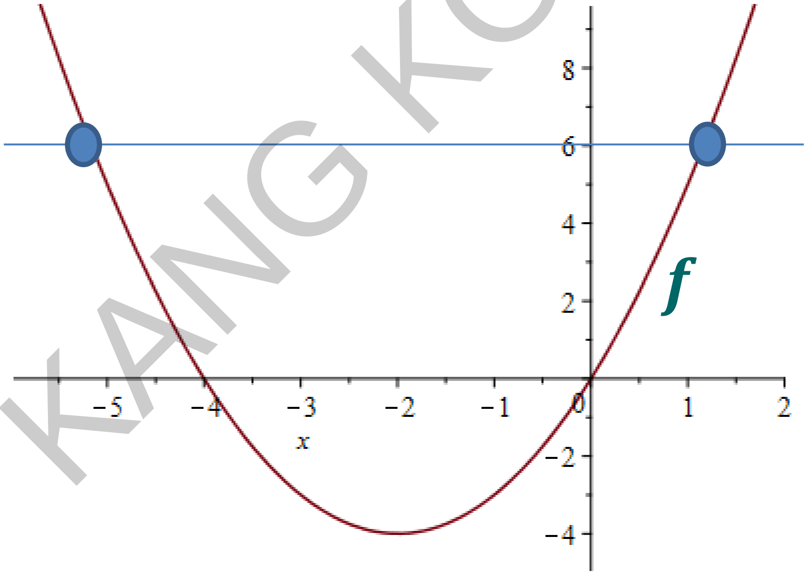
# Solution

(2)(a) A horizontal line intersects the graph at two points, in particular $f(0) = f(-4) = 0$. Therefore, $f$ is not a one-to-one function.





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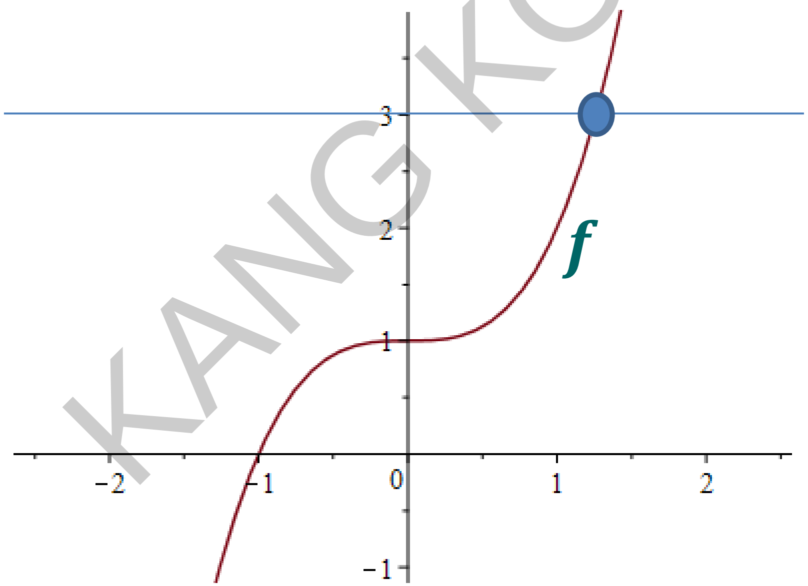
# Solution

(2)(b) A horizontal line intersects the graph at only one point.
Therefore, $f$ is a one-to-one function.








Bloom: Understanding

# Self-check

(1) By using the algebraic method, determine whether $f$ is a one-to-one function or not.

(a) $f(x) = 3 - 4x$

(b) $f(x) = |2x + 5|$

(2) Use graphical method to determine whether each of the following functions is a one-to-one function.

(a) $f(x) = x^2 - 9$

(b) $f(x) = x^3$

Bloom: Applying

# Answer Self-check

**(1) (a) One-to-one** **(b) Not one-to-one**

**(2) (a) Not one-to-one** **(b) One-to-one**



*Bloom: Applying*

# Domain and Range of a Function

Domain of a function $f$ is the set of all real values of $x$ for which function $f$ is defined as a real number.
In a graph of $y = f(x)$, the domain is shown on the $x - axis$.

Domain of a function $f$ is the set of all real values of $y$ for which function $f$ is defined as a real number.
In a graph of $y = f(x)$, the domain is shown on the $y - axis$.



**Two methods to determine domain and range.**
**Method 1: Graphical method**
**Method 2: Algebraic method**

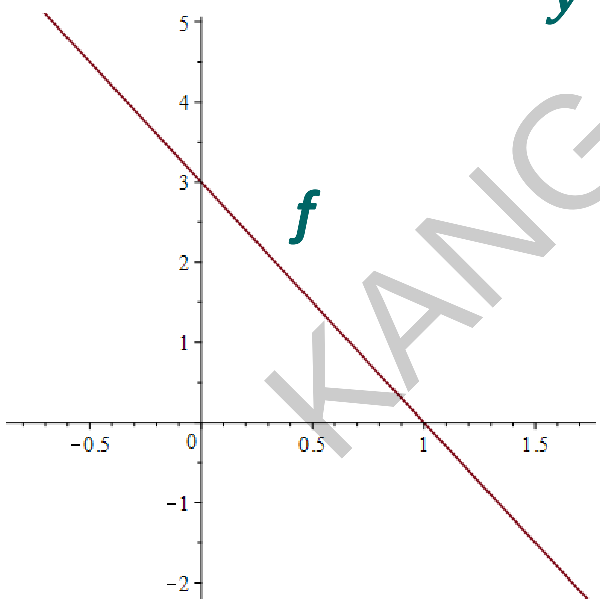
# Linear Graph

$f(x) = 3 - 3x$

$x - intercept: When\ y = 0$
$x = 1$

$y - intercept: When\ x = 0$
$y = 3$

Method 1: Graphical method to determine domain and range.




Domain of $f = (-\infty, \infty)$

Range of $f = (-\infty, \infty)$

# Quadratic Graph

$$f(x) = x^2 - 2x - 3$$

$y - intercept$: When $x = 0$

$$y = -3$$

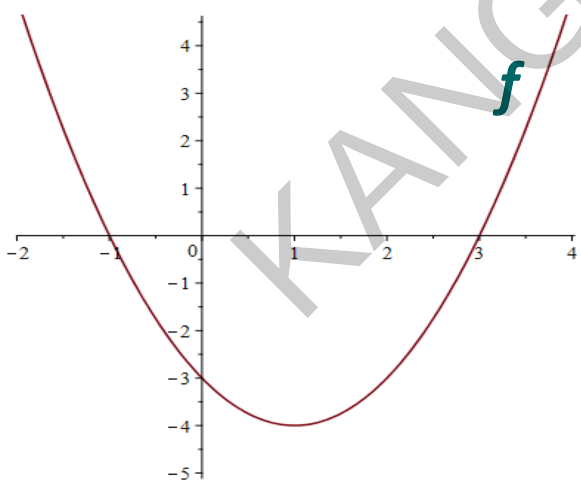
Method 1: Graphical method to determine domain and range.

$$f(x) = x^2 - 2x - 3$$

$$f(x) = (x - 1)^2 - 4$$

Completing the squares to get minimum point.

$\therefore$ *Minimum point is* $(1, -4)$.




Domain of $f = (-\infty, \infty)$

Range of $f = [-4, \infty)$

# Cubic Graph

$f(x) = (x+1)^3$

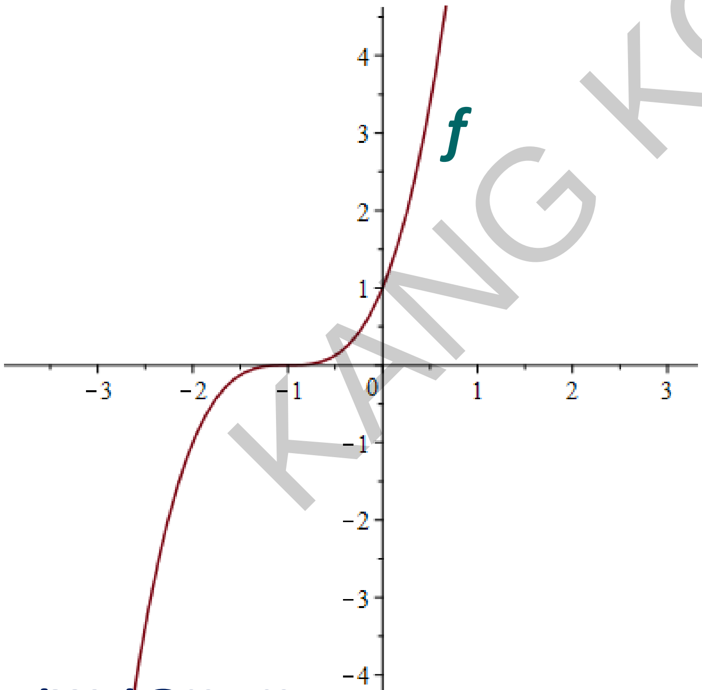
$x - intercept: When\ y = 0$

$x = -1$

$y - intercept: When\ x = 0$

$y = 1$

Method 1: Graphical method to determine domain and range.




Domain of $f = (-\infty, \infty)$

Range of $f = (-\infty, \infty)$



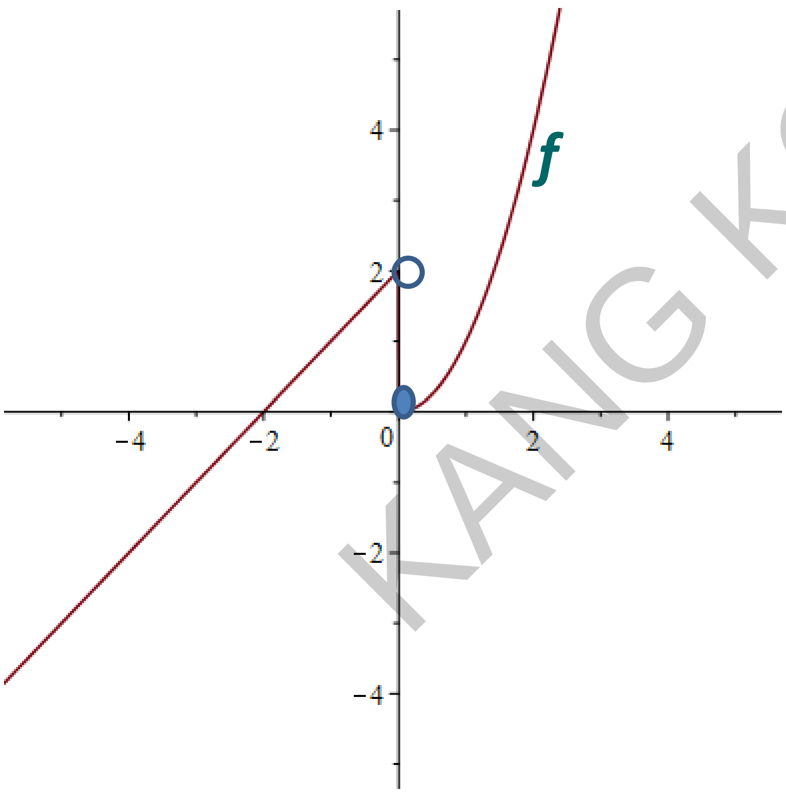


Bloom: Remembering

# Piecewise Functions Graph

$$f(x) = \begin{cases} x^2, & x \geq 0 \\ x + 2, & x < 0 \end{cases}$$

**Method 1: Graphical method to determine domain and range.**




**Domain of** $f = (-\infty, \infty)$

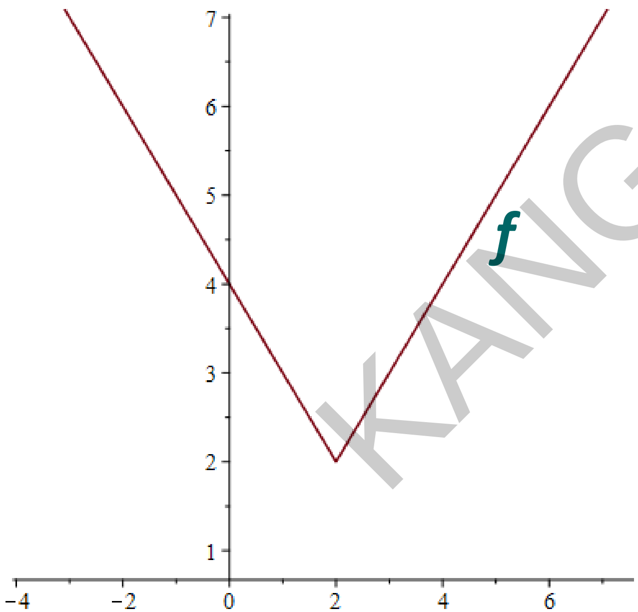
**Range of** $f = (-\infty, \infty)$

# Absolute Value Functions Graph

$f(x) = |2 - x| + 2$

$$f(x) = \begin{cases} (2-x)+2, & 2-x \geq 0 \\ -(2-x)+2, & 2-x < 0 \end{cases}$$

$$f(x) = \begin{cases} 4-x, & x \leq 2 \\ x, & x > 2 \end{cases}$$




**Method 1: Graphical method to determine domain and range.**

**Domain of** $f = (-\infty, \infty)$

**Range of** $f = [2, \infty)$

# Rational Functions Graph

$$f(x) = \frac{1}{x} + 3$$

Method 2: Algebraic method to determine domain and range.

**To determine domain:**

$x \neq 0$ — For f(x) to be defined.

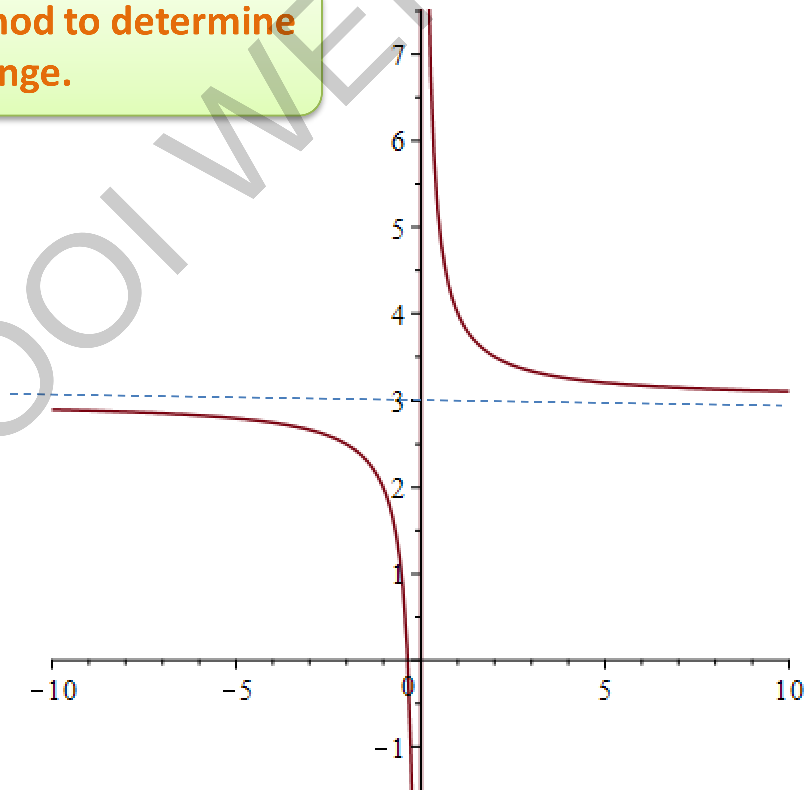
**To determine range:**

$$y = \frac{1}{x} + 3$$ Letting $y = f(x)$.

$$y - 3 = \frac{1}{x}$$

$$x = \frac{1}{y-3}$$ Solving for x in terms of y.

$y \neq 3$ — For $\frac{1}{y-3}$ to be defined.




Domain of $f = (-\infty, 0) \cup (0, \infty)$

Range of $f = (-\infty, 3) \cup (3, \infty)$

# Square Root Functions Graph

$f(x) = \sqrt{x-2}$

Method 2: Algebraic method to determine domain and range.

**To determine domain:**

$x - 2 \geq 0$

$x \geq 2$

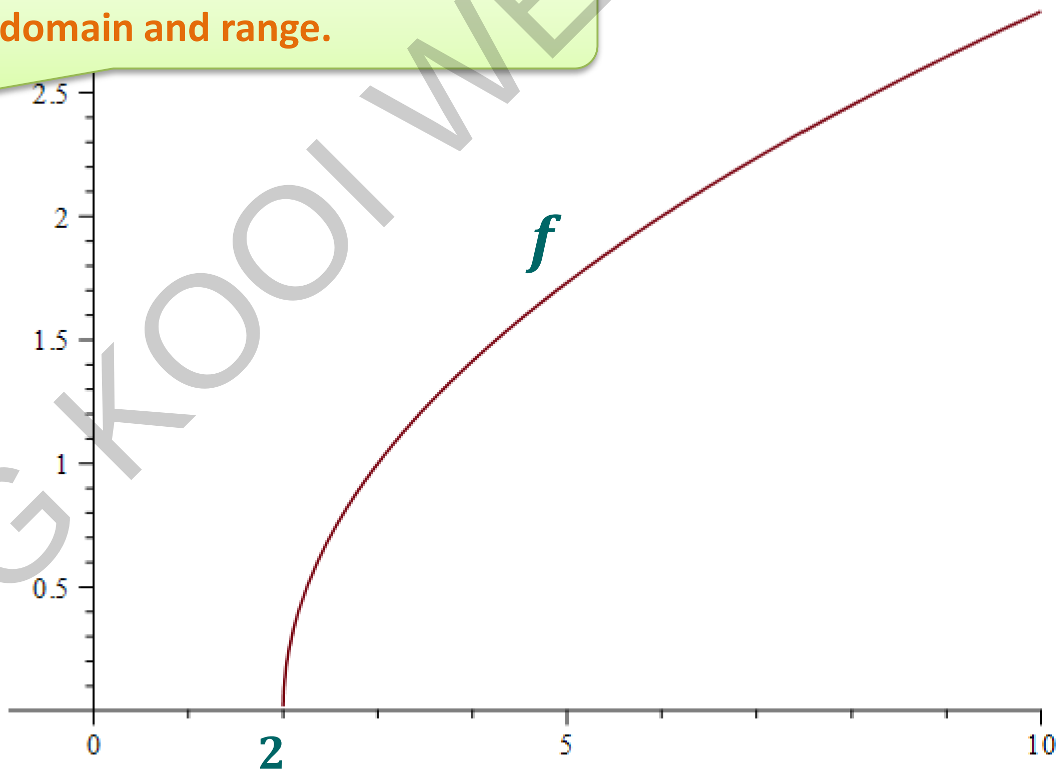
**Domain of** $f = [2, \infty)$

**To determine range:**

$\sqrt{x-2} \geq 0$

$f(x) \geq 0$

**Range of** $f = [0, \infty)$



# Example

(1) A function is defined by $\boldsymbol{f: x \rightarrow x^2 + 5}$, $\boldsymbol{x \epsilon R}$.

Find

(a) $\boldsymbol{f(3)}$

(b) $\boldsymbol{f(-5)}$

(c) $\boldsymbol{f(x-1)}$





*Bloom: Understanding*

# Example

(2) Find the domain and range of the following functions.

(a) $f(x) = 2x^2 - 4x + 1$

(b) $f(x) = \sqrt{x - 5}$

(c) $f(x) = \frac{1}{x}$





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# Solution

(1) $f: x \to x^2 + 5$

(a) $f(3) = (3)^2 + 5 = 14$

(b) $f(-5) = (-5)^2 + 5 = 30$

(c) $f(x-1) = (x-1)^2 + 5$

$= x^2 - 2x + 1 + 5$

$= x^2 - 2x + 6$





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# Solution

(2) (a) $f(x) = 2\left[x^2 - 2x + \frac{1}{2}\right]$

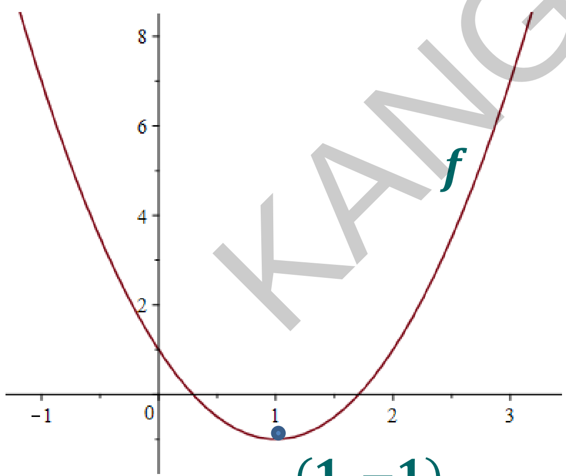
$= 2\left[(x-1)^2 - 1 + \frac{1}{2}\right]$

$= 2\left[(x-1)^2 - \frac{1}{2}\right]$

$= 2(x-1)^2 - 1$

Completing the square. Coefficient of $x^2$ must 1.

Minimum point is (1,-1).




$(1, -1)$

Real number.

$D_f = R$ $R_f = [-1, \infty)$

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# Solution

(2) (b) $f(x) = \sqrt{x-5}$

**To determine domain:**

$f(x)$ is only defined if $x - 5 \geq 0$

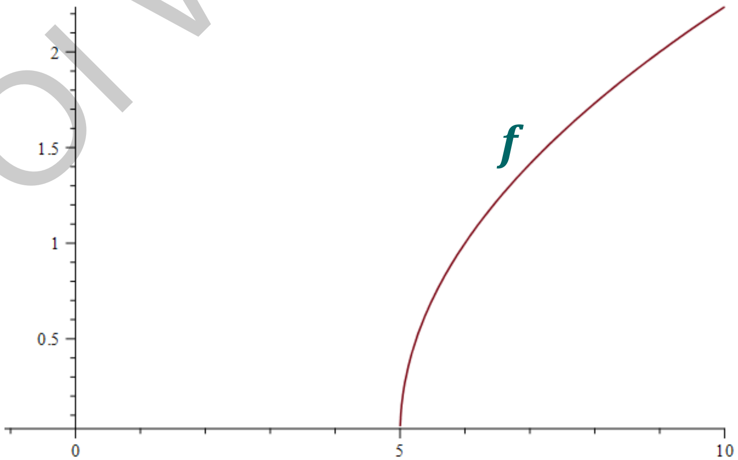
$\therefore x \geq 5$

$D_f = [5, \infty)$

**To determine range:**

$f(x) \geq 0$

$R_f = [0, \infty)$




$D_f = [5, \infty)$

$R_f = [0, \infty)$





Bloom: Understanding

# Solution

(2) (c) $f(x) = \dfrac{1}{x}$

**To determine domain:**

$f(x)$ is defined for all real numbers except 0.

$D_f = (-\infty, 0) \cup (0, \infty)$

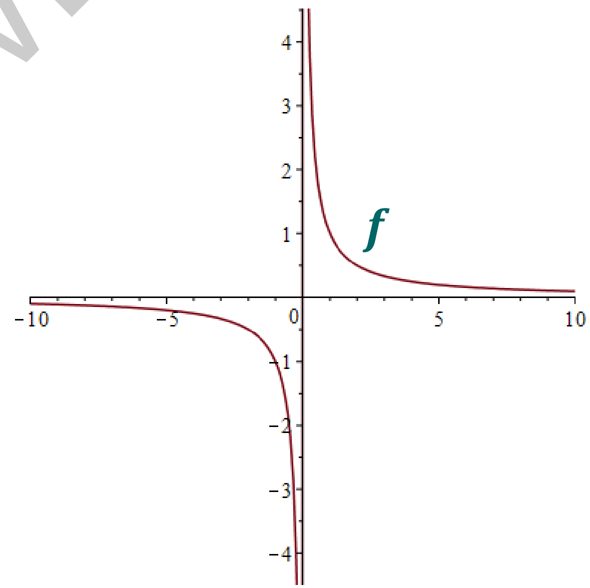
**To determine range:**

$y = \dfrac{1}{x}$ — Let $y = f(x)$.

$x = \dfrac{1}{y}$ — Rearrange to let x in terms of y.

$y \neq 0$ — For $\dfrac{1}{y}$ to be defined.

$R_f = (-\infty, 0) \cup (0, \infty)$




$D_f = (-\infty, 0) \cup (0, \infty)$

$R_f = (-\infty, 0) \cup (0, \infty)$

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# Self-check

(1) The linear function $f$ is defined by $f: x \to 5x - 3, x \epsilon R$. Find

(a) $f\left(\frac{1}{3}\right)$

(b) the value of $x$ for which $f(x) = x$ .

(2) Find the domain and range of the following functions

(a) $f(x) = 3x^2 - 7x - 1$

(b) $f(x) = \sqrt{2x - 1}$

(c) $f(x) = \frac{1}{x - 3}$





Bloom: Applying

# Answer Self-check

(1) (a) $-\frac{4}{3}$ (b) $\frac{3}{4}$

(2) (a) $D_f = R, R_f = \left[-\frac{61}{12}, \infty\right)$

(b) $D_f = \left[\frac{1}{2}, \infty\right), R_f = [0, \infty)$

(c) $D_f = (-\infty, 3) \cup (3, \infty)$

$R_f = (-\infty, 0) \cup (0, \infty)$

Bloom: Applying

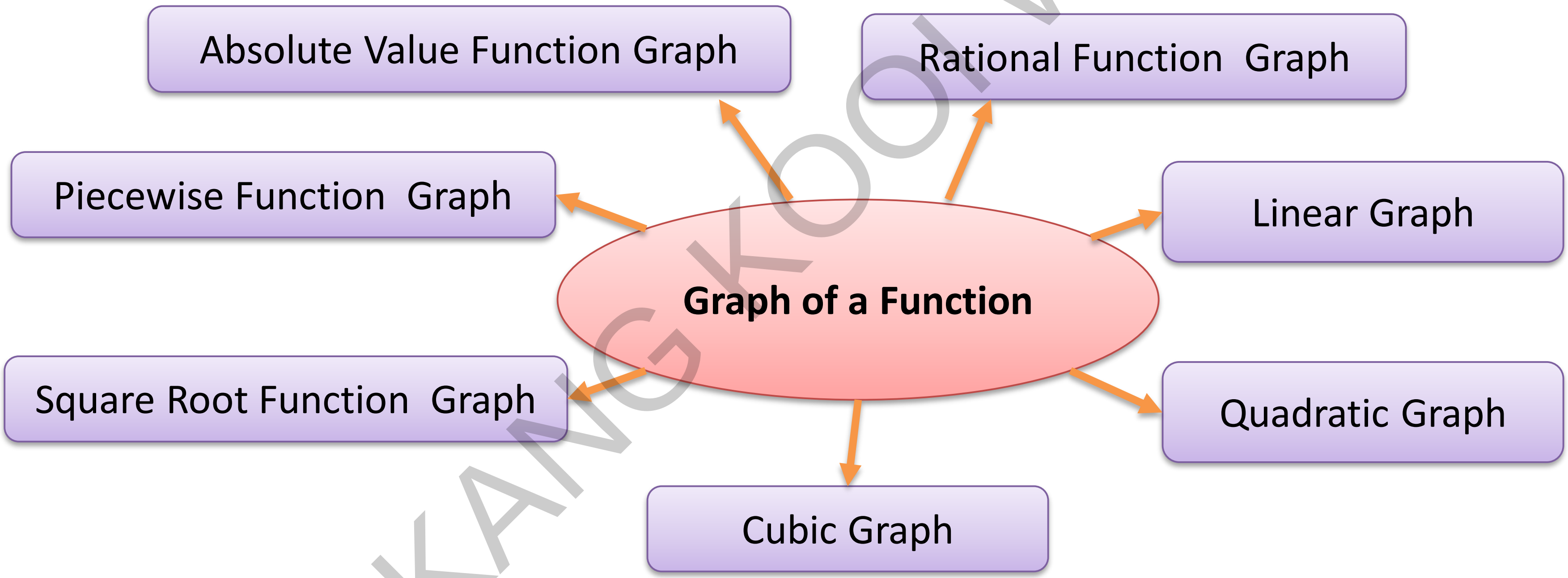
# Summary

Horizontal line test → Identify one-to-one function.

Vertical line test → Identify a function.

**Algebraic or Graphical Approach to determine Domain and Range.**
**Sometime can mix both method to determine Domain and Range.**

# Summary


Absolute Value Function Graph
Rational Function Graph
Piecewise Function Graph
Linear Graph
Graph of a Function
Square Root Function Graph
Quadratic Graph
Cubic Graph

# Key Terms

- Functions
- Domain
- Range
- One-to-one function
- Horizontal line test
- Vertical line test
- Graph