



**KEMENTERIAN
PENDIDIKAN
MALAYSIA**

QS 015/2

**Matriculation Programme
Examination**

Semester 1

Session 2012/2013

1. Given that $f(x) = \begin{cases} 1 + e^x, & x < 1 \\ 1, & x = 1 \\ 2 - x, & x > 1 \end{cases}$

Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$. Does the $\lim_{x \rightarrow 1} f(x)$ exist? State your reason.

2. Prove that $1 + \tan 2\theta \tan \theta = \sec 2\theta$.

3. Find the following limits:

(a) $\lim_{x \rightarrow \infty} \frac{2x^2 + x - 4}{1 - x^2}$

(b) $\lim_{x \rightarrow 2} \frac{3 - \sqrt{x+7}}{x^2 - 4}$

4. Express $\frac{2x^3 - 7x^2 + 17x - 19}{2x^2 - 7x + 6}$ in the form of partial fractions.

5. (a) Given that $f(x) = \begin{cases} \frac{|x^2 - x - 2|}{x^2 - 2x}, & x \neq 0, 2 \\ 0, & x = 2 \end{cases}$

Find the $\lim_{x \rightarrow 2} f(x)$. Is $f(x)$ continuous at $x = 2$?

(b) A function $f(x)$ is defined by $f(x) = \begin{cases} ax + 6, & x < 4 \\ x^2 + 2, & 4 \leq x < 6 \\ 2 - \beta x, & x \geq 6 \end{cases}$

Determine the values of the constants α and β if $f(x)$ is continuous.

6. The polynomial $P(x) = 2x^3 + ax^2 + bx - 24$ has a factor $(x - 2)$ and a remainder 15 when divided by $(x + 3)$.

(a) Find the values of a and b .

(b) Factorise $P(x)$ completely and find all zeroes of $P(x)$.

7. Given $f(\theta) = 3 \sin \theta - 2 \cos \theta$.

(a) Express $f(\theta)$ in the form of $R \sin(\theta - \alpha)$, where $R > 0$, $0 \leq \alpha \leq \frac{\pi}{2}$.
Hence, find the maximum and minimum values of $f(\theta)$.

(b) Solve $f(\theta) = \sqrt{\frac{3}{2}}$ for $0^\circ \leq \theta \leq 360^\circ$.

8. (a) Given that $y = \frac{1}{\sqrt{2x+1}}$.
- By using the first principle of derivative, find $\frac{dy}{dx}$.
 - Find $\frac{d^2y}{dx^2}$.
- (b) Find $\frac{dy}{dx}$ of the following:
- $y = e^{2x} \tan x$
 - $y = x^{\sec x}$
9. (a) A conical tank is of height 12m and surface diameter 8m. Water is pumped into the tank at the rate of $50 \text{ m}^3/\text{min}$. How fast is the water level increasing when the depth of the water is 6m?
- (b) A cylindrical container of radius r and height h has a constant volume V . The cost of the materials for the surface of both of its ends is twice the cost of its sides. State h in terms of r and V . Hence, find h and r in terms of V such that the cost is minimum.
10. (a) Given $3y^2 - xy + x^2 = 3$. By using implicit differentiation,
- Find the values of $\frac{dy}{dx}$ at $x = 1$.
 - Show that $(6y - x) \frac{d^2y}{dx^2} + 6 \left(\frac{dy}{dx}\right)^2 - 2 \frac{dy}{dx} + 2 = 0$.
- (b) Consider the parametric equations
- $$x = 3t - \frac{2}{t}, \quad y = 3t + \frac{2}{t} \quad \text{where } t \neq 0.$$
- Show that $\frac{dy}{dx} = 1 - \frac{4}{3t^2+2}$.
 - Find $\frac{d^2y}{dx^2}$ when $t = 1$.

END OF QUESTION PAPER

1. Given that $f(x) = \begin{cases} 1 + e^x, & x < 1 \\ 1, & x = 1 \\ 2 - x, & x > 1 \end{cases}$

Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$. Does the $\lim_{x \rightarrow 1} f(x)$ exist? State your reason.

SOLUTION

$$f(x) = \begin{cases} 1 + e^x, & x < 1 \\ 1, & x = 1 \\ 2 - x, & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (1 + e^x)$$

$$= (1 + e^1)$$

$$= 1 + e$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2 - x)$$

$$= 2 - 1$$

$$= 1$$

Since $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$. Therefore $\lim_{x \rightarrow 1} f(x)$ does not exist.

2. Prove that $1 + \tan 2\theta \tan \theta = \sec 2\theta$.

SOLUTION

$$\begin{aligned}1 + \tan 2\theta \tan \theta &= 1 + \left(\frac{\sin 2\theta}{\cos 2\theta}\right) \left(\frac{\sin \theta}{\cos \theta}\right) \\&= 1 + \left(\frac{\sin 2\theta \sin \theta}{\cos 2\theta \cos \theta}\right) \\&= \frac{\cos 2\theta \cos \theta + \sin 2\theta \sin \theta}{\cos 2\theta \cos \theta} \\&= \frac{\cos(2\theta - \theta)}{\cos 2\theta \cos \theta} \\&= \frac{\cos(\theta)}{\cos 2\theta \cos \theta} \\&= \frac{1}{\cos 2\theta} \\&= \sec 2\theta.\end{aligned}$$

$$\begin{aligned}\cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \cos(2\theta - \theta) &= \cos 2\theta \cos \theta + \sin 2\theta \sin \theta\end{aligned}$$

3. Find the following limits:

$$(a) \lim_{x \rightarrow \infty} \frac{2x^2 + x - 4}{1 - x^2}$$

$$(b) \lim_{x \rightarrow 2} \frac{3 - \sqrt{x+7}}{x^2 - 4}$$

SOLUTION

3(a)

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{2x^2 + x - 4}{1 - x^2} &= \lim_{x \rightarrow \infty} \frac{\frac{2x^2 + x - 4}{x^2}}{\frac{1 - x^2}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x} - \frac{4}{x^2}}{\frac{1}{x^2} - 1} \\ &= \frac{2 + 0 - 0}{0 - 1} \\ &= -2 \end{aligned}$$

3(b)

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{3 - \sqrt{x+7}}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{3 - \sqrt{x+7}}{x^2 - 4} \cdot \frac{3 + \sqrt{x+7}}{3 + \sqrt{x+7}} \\ &= \lim_{x \rightarrow 2} \frac{9 - (x+7)}{(x+2)(x-2)(3 + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{2 - x}{(x+2)(x-2)(3 + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{-(x-2)}{(x+2)(x-2)(3 + \sqrt{x+7})} \\ &= \lim_{x \rightarrow 2} \frac{-1}{(x+2)(3 + \sqrt{x+7})} \end{aligned}$$

$$= \frac{-1}{(2+2)(3+\sqrt{2+7})}$$

$$= \frac{-1}{(4)(3+\sqrt{9})}$$

$$= \frac{-1}{24}$$

KANG KOOI WEI

4. Express $\frac{2x^3-7x^2+17x-19}{2x^2-7x+6}$ in the form of partial fractions.

SOLUTION

$$\frac{2x^3-7x^2+17x-19}{2x^2-7x+6} \rightarrow \text{Improper fraction}$$

$$\begin{array}{r} x \\ 2x^2 - 7x + 6 \overline{) 2x^3 - 7x^2 + 17x - 19} \\ \underline{2x^3 - 7x^2 + 6x} \\ 11x - 19 \end{array}$$

$$\frac{2x^3 - 7x^2 + 17x - 19}{2x^2 - 7x + 6} = x + \frac{11x - 19}{2x^2 - 7x + 6}$$

$$\frac{2x^3 - 7x^2 + 17x - 19}{2x^2 - 7x + 6} = x + \frac{11x - 19}{(2x - 3)(x - 2)}$$

$$\frac{11x - 19}{(2x - 3)(x - 2)} = \frac{A}{(2x - 3)} + \frac{B}{(x - 2)}$$

$$\frac{11x - 19}{(2x - 3)(x - 2)} = \frac{A(x - 2) + B(2x - 3)}{(2x - 3)(x - 2)}$$

$$11x - 19 = A(x - 2) + B(2x - 3)$$

When $x = 2$

$$11(2) - 19 = A[(2) - 2] + B[(2(2) - 3)]$$

$$3 = A[0] + B[(1)]$$

$$B = 3$$

When $x = \frac{3}{2}$

$$11\left(\frac{3}{2}\right) - 19 = A\left[\left(\frac{3}{2}\right) - 2\right] + B\left[2\left(\frac{3}{2}\right) - 3\right]$$

$$-\frac{5}{2} = A\left[-\frac{1}{2}\right] + B[(0)]$$

$$A = 5$$

$$\frac{11x - 19}{(2x - 3)(x - 2)} = \frac{5}{(2x - 3)} + \frac{3}{(x - 2)}$$

$$\frac{2x^3 - 7x^2 + 17x - 19}{2x^2 - 7x + 6} = x + \frac{11x - 19}{(2x - 3)(x - 2)}$$

$$\frac{2x^3 - 7x^2 + 17x - 19}{2x^2 - 7x + 6} = x + \frac{5}{(2x - 3)} + \frac{3}{(x - 2)}$$

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5. (a) Given that $f(x) = \begin{cases} \frac{|x^2-x-2|}{x^2-2x}, & x \neq 0, 2 \\ 0, & x = 2 \end{cases}$

Find the $\lim_{x \rightarrow 2} f(x)$. Is $f(x)$ continuous at $x = 2$?

(b) A function $f(x)$ is defined by $f(x) = \begin{cases} ax + 6, & x < 4 \\ x^2 + 2, & 4 \leq x < 6 \\ 2 - \beta x, & x \geq 6 \end{cases}$

Determine the values of the constants α and β if $f(x)$ is continuous.

SOLUTION

5(a)

$$f(x) = \begin{cases} \frac{|x^2 - x - 2|}{x^2 - 2x}, & x \neq 0, 2 \\ 0, & x = 2 \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \frac{|x^2 - x - 2|}{x^2 - 2x} \\ &= \lim_{x \rightarrow 2^-} \frac{|(x+1)(x-2)|}{x(x-2)} \\ &= \lim_{x \rightarrow 2^-} \frac{(x+1)[-(x-2)]}{x(x-2)} \\ &= \lim_{x \rightarrow 2^-} \frac{-(x+1)}{x} \\ &= \frac{-(2+1)}{2} \\ &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} \frac{|x^2 - x - 2|}{x^2 - 2x} \\ &= \lim_{x \rightarrow 2^+} \frac{|(x+1)(x-2)|}{x(x-2)} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{x \rightarrow 2^+} \frac{(x+1)(x-2)}{x(x-2)} \\
 &= \lim_{x \rightarrow 2^+} \frac{x+1}{x} \\
 &= \frac{2+1}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$
 $\lim_{x \rightarrow 2} f(x)$ does not exist
 $f(x)$ discontinuous at $x = 2$

5(b)

$$f(x) = \begin{cases} \alpha x + 6 & , \quad x < 4 \\ x^2 + 2 & , \quad 4 \leq x < 6 \\ 2 - \beta x & , \quad x \geq 6 \end{cases}$$

f is continuous at $x = 4$

$\lim_{x \rightarrow 4} f(x)$ exists.

$$\lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^+} f(x)$$

$$\lim_{x \rightarrow 4^-} \alpha x + 6 = \lim_{x \rightarrow 4^+} x^2 + 2$$

$$\alpha(4) + 6 = (4)^2 + 2$$

$$4\alpha = 12$$

$$\alpha = 3$$

f is continuous at $x = 6$

$\lim_{x \rightarrow 6} f(x)$ exists.

$$\lim_{x \rightarrow 6^-} f(x) = \lim_{x \rightarrow 6^+} f(x)$$

$$\lim_{x \rightarrow 6^-} x^2 + 2 = \lim_{x \rightarrow 6^+} 2 - \beta x$$

$$(6)^2 + 2 = 2 - \beta(6)$$

$$38 = 2 - 6\beta$$

$$6\beta = -36$$

$$\beta = -6$$

$$\therefore \alpha = 3, \quad \beta = -6$$

KANG KOOI WEI

6. The polynomial $P(x) = 2x^3 + ax^2 + bx - 24$ has a factor $(x - 2)$ and a remainder 15 when divided by $(x + 3)$.
- Find the values of a and b .
 - Factorise $P(x)$ completely and find all zeroes of $P(x)$.

SOLUTION**6(a)**

$$P(2) = 0$$

$$P(-3) = 15$$

$$P(x) = 2x^3 + ax^2 + bx - 24$$

$$P(2) = 2(2)^3 + a(2)^2 + b(2) - 24 = 0$$

$$16 + 4a + 2b - 24 = 0$$

$$4a + 2b = 8 \dots\dots\dots(1)$$

$$P(-3) = 2(-3)^3 + a(-3)^2 + b(-3) - 24 = 15$$

$$-54 + 9a - 3b - 24 = 15$$

$$9a - 3b = 93$$

$$3a - b = 31 \dots\dots\dots(2)$$

$$(2) \times 2$$

$$6a - 2b = 62 \dots\dots\dots(3)$$

$$(1)+(2)$$

$$10a = 70$$

$$a = 7$$

$$b = -10$$

6(b)

$$P(x) = 2x^3 + 7x^2 - 10x - 24$$

$$= (x - 2)Q(x)$$

$$\begin{array}{r}
 2x^2 + 11x + 12 \\
 x - 2 \overline{) 2x^3 + 7x^2 - 10x - 24} \\
 \underline{2x^3 - 4x^2} \\
 11x^2 - 10x - 24 \\
 \underline{11x^2 - 22x} \\
 12x - 24 \\
 \underline{12x - 24} \\
 0
 \end{array}$$

$$P(x) = (x - 2)(2x^2 + 11x + 12)$$

$$= (x - 2)(2x + 3)(x + 4)$$

when $P(x) = 0$

$$(x - 2)(2x + 3)(x + 4) = 0$$

$$x = 2, \quad x = -\frac{3}{2}, \quad x = -4$$

The zeroes are $2, -\frac{3}{2}$ and -4

7. Given $f(\theta) = 3 \sin \theta - 2 \cos \theta$.
- Express $f(\theta)$ in the form of $R \sin(\theta - \alpha)$, where $R > 0$, $0 \leq \alpha \leq \frac{\pi}{2}$.
Hence, find the maximum and minimum values of $f(\theta)$.
 - Solve $f(\theta) = \sqrt{\frac{3}{2}}$ for $0^\circ \leq \theta \leq 360^\circ$.

SOLUTION

$$\begin{aligned} \text{a) } 3 \sin \theta - 2 \cos \theta &= R \sin(\theta - \alpha) \\ &= R \sin \theta \cos \alpha - R \cos \theta \sin \alpha \end{aligned}$$

$$R \cos \alpha = 3 \quad (1)$$

$$R \sin \alpha = 2 \quad (2)$$

$$(1)^2 + (2)^2:$$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 3^2 + 2^2$$

$$R^2(1) = 13$$

$$R = \sqrt{13}$$

$$(2) \div (1):$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{2}{3}$$

$$\tan \alpha = \frac{2}{3}$$

$$\alpha = 0.588$$

$$3 \sin \theta - 2 \cos \theta = \sqrt{13} \sin(\theta - 0.588)$$

$$-1 \leq \sin(\theta - 0.588) \leq 1$$

$$-\sqrt{13} \leq \sqrt{13} \sin(\theta - 0.588) \leq \sqrt{13}$$

$$-\sqrt{13} \leq 3 \sin \theta - 2 \cos \theta \leq \sqrt{13}$$

$$-\sqrt{13} \leq f(\theta) \leq \sqrt{13}$$

$$\text{Minimum value of } f(\theta) = -\sqrt{13}$$

$$\text{Maximum value of } f(\theta) = \sqrt{13}$$

$$\text{b) } f(\theta) = \sqrt{\frac{13}{2}}$$

$$\sqrt{13} \sin(\theta - 0.588) = \sqrt{\frac{13}{2}}$$

$$\sin(\theta - 0.588) = \frac{1}{\sqrt{2}}$$

$$\theta - 0.588 = 0.785, \pi - 0.785$$

$$\theta = 1.373, 2.945$$

$$\theta = 78.7^\circ, 168.7^\circ$$

KANG KOOI WEI

8. (a) Given that $y = \frac{1}{\sqrt{2x+1}}$.
- By using the first principle of derivative, find $\frac{dy}{dx}$.
 - Find $\frac{d^2y}{dx^2}$.
- (b) Find $\frac{dy}{dx}$ of the following:
- $y = e^{2x} \tan x$
 - $y = x^{\sec x}$

SOLUTION

(a) $y = \frac{1}{\sqrt{2x+1}}$

i) Let $f(x) = \frac{1}{\sqrt{2x+1}}$

$$f(x+h) = \frac{1}{\sqrt{2(x+h)+1}}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{2(x+h)+1}} - \frac{1}{\sqrt{2x+1}}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{2x+1} - \sqrt{2(x+h)+1}}{\sqrt{2(x+h)+1}\sqrt{2x+1}}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{2x+1} - \sqrt{2x+2h+1}}{h\sqrt{2(x+h)+1}\sqrt{2x+1}} \cdot \frac{\sqrt{2x+1} + \sqrt{2x+2h+1}}{\sqrt{2x+1} + \sqrt{2x+2h+1}}$$

$$= \lim_{h \rightarrow 0} \frac{(2x+1) - (2x+2h+1)}{h\sqrt{2(x+h)+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})}$$

$$= \lim_{h \rightarrow 0} \frac{-2h}{h\sqrt{2(x+h)+1}\sqrt{2x+1}(\sqrt{2x+1} + \sqrt{2x+2h+1})}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{-2}{\sqrt{2(x+h)+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+2h+1})} \\
 &= \frac{-2}{\sqrt{2x+1}\sqrt{2x+1}(\sqrt{2x+1}+\sqrt{2x+1})} \\
 &= \frac{-2}{(2x+1)(2\sqrt{2x+1})} \\
 &= -\frac{1}{(2x+1)^{\frac{3}{2}}}
 \end{aligned}$$

$$\text{ii) } \frac{dy}{dx} = -(2x+1)^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}(2x+1)^{-\frac{5}{2}}(2)$$

$$= \frac{3}{(2x+1)^{\frac{5}{2}}}$$

$$\text{b i) } y = e^{2x} \tan x$$

$$\begin{aligned}
 \frac{dy}{dx} &= e^{2x}(\sec^2 x) + \tan x(2e^{2x}) \\
 &= e^{2x}(\sec^2 x + 2 \tan x)
 \end{aligned}$$

$$\text{ii) } y = x^{\sec x}$$

$$\ln y = \ln x^{\sec x}$$

$$\ln y = \sec x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \sec x \left(\frac{1}{x} \right) + (\ln x)(\sec x \tan x)$$

$$\frac{dy}{dx} = y \left[\frac{\sec x + x(\ln x) \sec x \tan x}{x} \right]$$

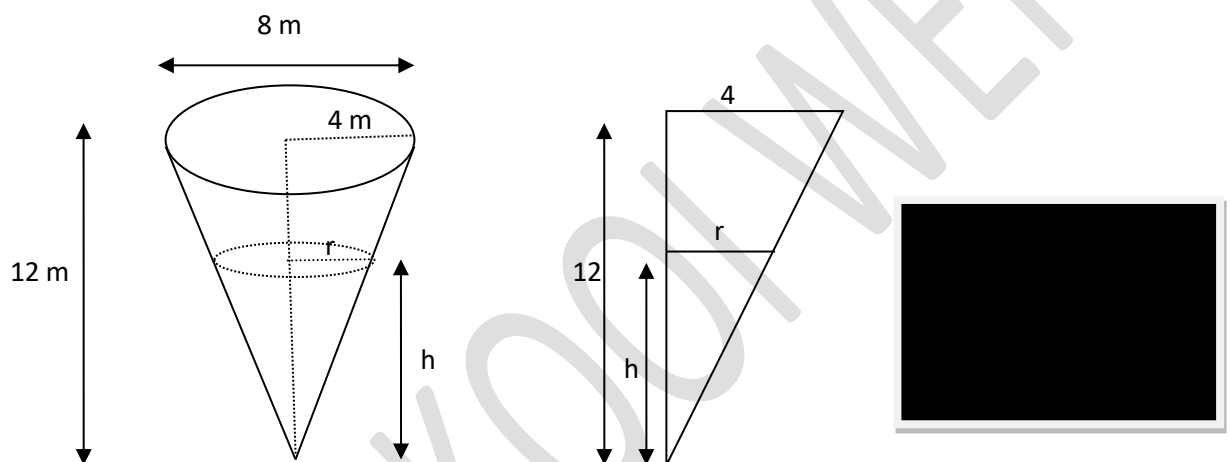
$$= x^{\sec x} \sec x \left[\frac{1 + x(\ln x) \tan x}{x} \right]$$

KANG KOOI WEI

9. (a) A conical tank is of height 12m and surface diameter 8m. Water is pumped into the tank at the rate of $50 \text{ m}^3/\text{min}$. How fast is the water level increasing when the depth of the water is 6m?
- (b) A cylindrical container of radius r and height h has a constant volume V . The cost of the materials for the surface of both of its ends is twice the cost of its sides. State h in terms of r and V . Hence, find h and r in terms of V such that the cost is minimum.

SOLUTION

9(a)



$$\frac{dV}{dt} = 50 \text{ m}^3/\text{min}$$

$$\left[\frac{dh}{dt} = ? \text{ when } h = 6 \right]$$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3} \pi \left(\frac{h}{3} \right)^2 h$$

$$V = \frac{1}{3} \pi \frac{h^3}{9}$$

$$V = \frac{1}{27} \pi h^3$$

$$\frac{dV}{dh} = \frac{1}{9} \pi h^2 = \frac{\pi h^2}{9}$$

$$\frac{dh}{dV} = \frac{9}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$$

$$\frac{dh}{dt} = \left(\frac{9}{\pi h^2} \right) \cdot (50)$$

$$\frac{dh}{dt} = \frac{450}{\pi h^2}$$

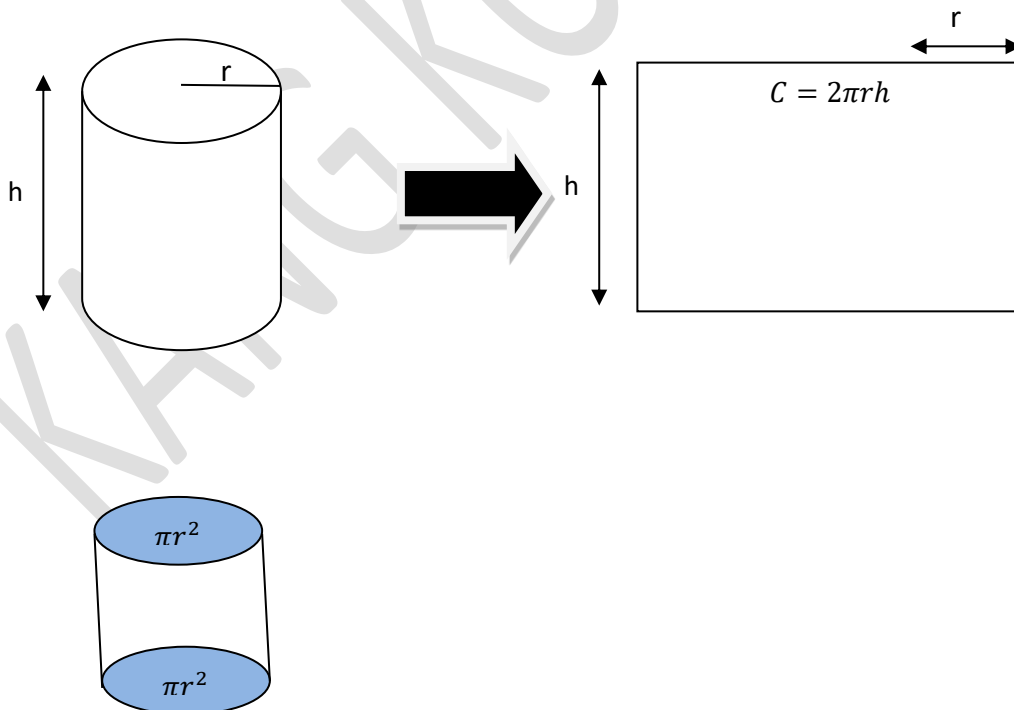
When $h = 6$

$$\frac{dh}{dt} = \frac{450}{\pi(6)^2}$$

$$\frac{dh}{dt} = \frac{450}{36\pi}$$

$$\frac{dh}{dt} = \frac{25}{2\pi} \text{ m/min}$$

9(b)



$$V = \pi r^2 h$$

$$\therefore h = \frac{V}{\pi r^2}$$

The cost,

$$\begin{aligned} C &= 2\pi r^2(2) + 2\pi rh(1) \\ &= 4\pi r^2 + 2\pi r \left(\frac{V}{\pi r^2} \right) \end{aligned}$$

$$C = 4\pi r^2 + \frac{2V}{r}$$

$$\frac{dC}{dr} = 8\pi r - \frac{2V}{r^2}$$

$$\text{when } \frac{dC}{dr} = 0$$

$$8\pi r - \frac{2V}{r^2} = 0$$

$$8\pi r = \frac{2V}{r^2}$$

$$r^3 = \frac{V}{4\pi}$$

$$r = \left(\frac{V}{4\pi} \right)^{\frac{1}{3}}$$

$$\frac{d^2C}{dr^2} = 8\pi + \frac{4V}{r^3}$$

$$\text{when } r = \left(\frac{V}{4\pi} \right)^{\frac{1}{3}},$$

$$\frac{d^2C}{dr^2} = 8\pi + \frac{4V}{\frac{V}{4\pi}} = 24\pi > 0 \text{ (min)}$$

$$\therefore r = \left(\frac{V}{4\pi} \right)^{\frac{1}{3}}$$

$$\therefore h = \frac{V}{\pi r^2} = \frac{V}{\pi \left(\frac{V}{4\pi} \right)^{\frac{2}{3}}} = \left(\frac{16V}{\pi} \right)^{\frac{1}{3}}$$

10 (a) Given $3y^2 - xy + x^2 = 3$. By using implicit differentiation,

i. Find the values of $\frac{dy}{dx}$ at $x = 1$.

ii. Show that $(6y - x)\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 2 = 0$.

(b) Consider the parametric equations

$$x = 3t - \frac{2}{t}, \quad y = 3t + \frac{2}{t} \quad \text{where } t \neq 0.$$

iii. Show that $\frac{dy}{dx} = 1 - \frac{4}{3t^2+2}$.

iv. Find $\frac{d^2y}{dx^2}$ when $t = 1$.

SOLUTION

10[a(i)]

$$3y^2 - xy + x^2 = 3$$

$$6y \frac{dy}{dx} - \left[x \frac{dy}{dx} + y \right] + 2x = 0$$

$$6y \frac{dy}{dx} - x \frac{dy}{dx} - y + 2x = 0$$

$$(6y - x) \frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{6y - x}$$

When $x = 1$

$$3y^2 - (1)y + (1)^2 = 3$$

$$3y^2 - y + 1 = 3$$

$$3y^2 - y - 2 = 0$$

$$3y^2 - y - 2 = 0$$

$$(3y + 2)(y - 1) = 0$$

$$y = -\frac{2}{3} \text{ or } y = 1$$

$$6y \frac{dy}{dx} - x \left(\frac{dy}{dx} \right) + y(-1) + 2x = 0$$

$$\frac{dy}{dx} (6y - x) = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{6y - x}$$

For $x = 1$, $y = -\frac{2}{3}$:

$$\frac{dy}{dx} = \frac{\left(-\frac{2}{3}\right) - 2(1)}{6\left(-\frac{2}{3}\right) - 1} = \frac{8}{15}$$

For $x = 1$, $y = 1$:

$$\frac{dy}{dx} = \frac{1 - 2(1)}{6(1) - 1} = -\frac{1}{5}$$

10[a(ii)]

$$\frac{dy}{dx} (6y - x) = y - 2x$$

$$\frac{dy}{dx} \left(6 \frac{dy}{dx} - 1 \right) + (6y - x) \left(\frac{d^2 y}{dx^2} \right) = \frac{dy}{dx} - 2$$

$$6 \left(\frac{dy}{dx} \right)^2 - \frac{dy}{dx} + (6y - x) \left(\frac{d^2 y}{dx^2} \right) - \frac{dy}{dx} + 2 = 0$$

$$(6y - x) \frac{d^2 y}{dx^2} + 6 \left(\frac{dy}{dx} \right)^2 - 2 \frac{dy}{dx} + 2 = 0$$

10(b)

$$x = 3t - \frac{2}{t}, \quad y = 3t + \frac{2}{t}$$

i) $\frac{dx}{dt} = 3 + \frac{2}{t^2} = \frac{3t^2 + 2}{t^2}$

$$\frac{dy}{dt} = 3 - \frac{2}{t^2} = \frac{3t^2 - 2}{t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\begin{aligned} & \frac{3t^2 - 2}{t^2} \\ &= \frac{3t^2 + 2}{t^2} \\ &= \frac{3t^2 - 2}{3t^2 + 2} \end{aligned}$$

$$\begin{array}{r} 1 \\ 3t^2 + 2 \overline{) 3t^2 - 2} \\ \underline{3t^2 + 2} \\ -4 \end{array}$$

$$\frac{dy}{dx} = 1 - \frac{4}{3t^2 + 2}$$

$$\begin{aligned} \text{ii) } \frac{d}{dt} \left(\frac{dy}{dx} \right) &= \frac{d}{dt} \left(1 - \frac{4}{3t^2 + 2} \right) \\ &= \frac{d}{dt} [1 - 4(3t^2 + 2)^{-1}] \\ &= 4(3t^2 + 2)^{-2} (6t) \\ &= \frac{24t}{(3t^2 + 2)^2} \end{aligned}$$

$$\frac{d^2 y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{24t}{(3t^2 + 2)^2}}{\frac{3t^2 + 2}{t^2}} = \frac{24t^3}{(3t^2 + 2)^3}$$

when $t = 1$,

$$\frac{d^2 y}{dx^2} = \frac{24(1)^3}{(3(1)^2 + 2)^3} = \frac{24}{125}$$