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QS 015/2

Matriculation Programme Examination

Semester I

Session 2013/2014

1. Express $\frac{x^2}{x^2+3x+2}$ in partial fractions form.

2. State the values of R and α such that $3sin\theta + 6cos\theta = Rsin(\theta + \alpha)$ where R > 0 and $0° \leq \alpha < 90°$. Hence, solve $3sin\theta + 6cos\theta = \sqrt{5}$ for $0° \leq \theta < 180°$.

3. (a) Find the value of m if $\lim_{x\to 0} \frac{mx+3x^2}{4x-8x^2} = 3$.

   (b) Evaluate $\lim_{x\to 0} \frac{\sqrt{3-x}-\sqrt{3}}{x}$.

4. (a) Find $\frac{dy}{dx}$ if $y = cosec\{sin[ln(x+1)]\}$.

   (b) Obtain the second derivative of $y = \frac{\cos 3x}{e^{2x}}$ and express your answer in the simplest form.

5. A cubic polynomial $P(x)$ has remainders 3 and 1 when divided by $(x-1)$ and $(x-2)$, respectively.

   (a) Let $Q(x)$ be a linear factor such that $P(x) = (x-1)(x-2)Q(x) + \alpha x + \beta$, where α *and* β are constants. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.

   (b) Use the values of α *and* β from part (a) to determine $Q(x)$ if the coefficient of x^3 for $P(x)$ is 1 and $P(3) = 7$. Hence, solve for x if $P(x) = 7 - 3x$.

6. (a) State the definition of the continuity of a function at a point. Hence, find the value of d such that

$$f(x) = \begin{cases} e^{3x+d}, & x \leq 0 \\ 3x+5, & x > 0 \end{cases}$$

   Is continuous at $x = 0$.

   (b) A function f is defined by

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1 \end{cases}$$

   Determine the value(s) of k if f is:

   (i) Continuous for all $x \in \mathbb{R}$.

(ii) Differentiable for all $x \in \mathbb{R}$.

7. (a) Find the derivative of $f(x) = \frac{1}{x+1}$ by using the first principle.

(b) Use implicit differentiation to find:

(i) $\frac{dy}{dx}$ if $y\ln x = e^{x-y}$.

(ii) the value of $\frac{dy}{dx}$ if $\frac{1}{y} - \frac{1}{x} = 3$ when $x = \frac{1}{2}$.

8. A curve is defined by parametric equations

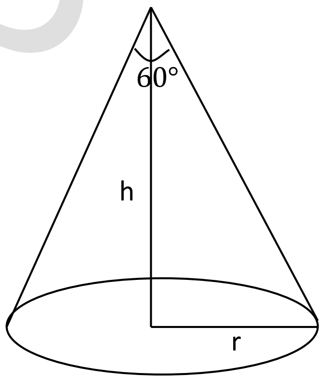
$$x = \ln(1+t), \qquad y = e^{t^2} \text{ for } t > -1.$$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

(b) Show that the curve has only one relative extremum at (0,1) and determine the nature of the point.

9. (a) A cylindrical container of volume $128\pi\, m^3$ is to be constructed with the same material for the top, bottom and lateral side. Find the dimensions of the container that will minimise the amount of the material needed.

(b) Gravel is poured onto a flat ground at the rate of $\frac{3}{20} m^3$ per minute to form a conical-shaped pile with vertex angle 60° as shown in the diagram below.



Compute the rate of chang of the height of the conical pile at the instant t = 10 minutes.

10. (a) Show that $\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta} = \cot\left[\frac{\beta-\alpha}{2}\right]$.

(b) Use trigonometric identities to verify that

(i) $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Hence, solve the equation $3\sin\theta + \cos\theta = 2$ for $0° \leq \theta \leq 180°$. Give your answers correct to three decimal places.

END OF QUESTION PAPER

1. Express $\frac{x^2}{x^2+3x+2}$ in partial fractions form.

SOLUTION

$$x^2+3x+2\,\overline{)\,x^2+0x+0}\quad 1$$
$$\underline{\qquad\qquad}$$
$$-3x-2$$

$$\frac{x^2}{x^2+3x+2}=1-\frac{3x+2}{x^2+3x+2}$$

$$=1-\frac{3x+2}{(x+1)(x+2)}$$

$$\frac{3x+2}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

$$=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$3x+2=A(x+2)+B(x+1)$$

When $x=-1$

$$3(-1)+2=A(-1+2)+B(-1+1)$$

$$-1=A(1)$$

$$A=-1$$

When $x=-2$

$$3(-2)+2=A(-2+2)+B(-2+1)$$

$$-4=B(-1)$$

$$B=4$$

$$\frac{3x+2}{(x+1)(x+2)}=\frac{-1}{(x+1)}+\frac{4}{(x+2)}$$

$$\frac{x^2}{x^2+3x+2}=1-\frac{3x+2}{(x+1)(x+2)}$$

$$=1-\left[\frac{-1}{(x+1)}+\frac{4}{(x+2)}\right]$$

$$=1+\frac{1}{(x+1)}-\frac{4}{(x+2)}$$

2. State the values of R and α such that $3sin\theta + 6cos\theta = Rsin(\theta + \alpha)$ where R > 0 and $0° \leq \alpha < 90°$. Hence, solve $3sin\theta + 6cos\theta = \sqrt{5}$ for $0° \leq \theta < 180°$.

SOLUTION

$$3sin\theta + 6cos\theta = Rsin(\theta + \alpha)$$

$$R = \sqrt{3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$$

$$\alpha = \tan^{-1}\frac{6}{3} = \tan^{-1} 2 = 63.4°$$

$$R = \sqrt{a^2 + b^2}$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

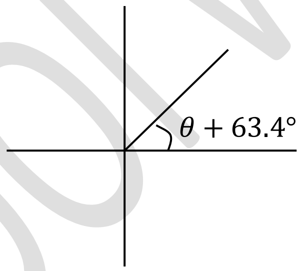
$$3sin\theta + 6cos\theta = \sqrt{5}$$

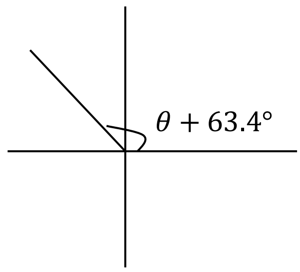
$$3\sqrt{5}sin(\theta + 63.4°) = \sqrt{5}$$

$$sin(\theta + 63.4°) = \frac{\sqrt{5}}{3\sqrt{5}}$$

$$sin(\theta + 63.4°) = \frac{1}{3}$$

$$\theta + 63.4° = \sin^{-1}\left(\frac{1}{3}\right) = 19.5°, 160.5°$$





$$\theta = 19.5° - 63.4°, \qquad 160.5° - 63.4°$$

$$\theta = -43.9°, \qquad 97.1°$$

Given that $0° \leq \theta < 180°$.

$$\therefore \theta = 97.1°$$

3. (a) Find the value of m if $\lim_{x\to 0}\frac{mx+3x^2}{4x-8x^2}=3$.

(b) Evaluate $\lim_{x\to 0}\frac{\sqrt{3-x}-\sqrt{3}}{x}$.

SOLUTION

3(a)

$$\lim_{x\to 0}\frac{mx+3x^2}{4x-8x^2}=3$$

$$\lim_{x\to 0}\frac{x(m+3x)}{x(4-8x)}=3$$

$$\lim_{x\to 0}\frac{(m+3x)}{(4-8x)}=3$$

$$\frac{m+3(0)}{4-8(0)}=3$$

$$\frac{m}{4}=3$$

$$m=12$$

3(b)

$$\lim_{x\to 0}\frac{\sqrt{3-x}-\sqrt{3}}{x}=\lim_{x\to 0}\frac{\sqrt{3-x}-\sqrt{3}}{x}\ X\ \frac{\sqrt{3-x}+\sqrt{3}}{\sqrt{3-x}+\sqrt{3}}$$

$$=\lim_{x\to 0}\frac{(3-x)-3}{x(\sqrt{3-x}+\sqrt{3})}$$

$$=\lim_{x\to 0}\frac{-x}{x(\sqrt{3-x}+\sqrt{3})}$$

$$=\lim_{x\to 0}\frac{-1}{\sqrt{3-x}+\sqrt{3}}$$

$$=\frac{-1}{\sqrt{3-0}+\sqrt{3}}$$

$$=\frac{-1}{2\sqrt{3}}$$

4. (a) Find $\frac{dy}{dx}$ if $y = cosec\{sin[ln(x+1)]\}$.

(b) Obtain the second derivative of $y = \frac{\cos 3x}{e^{2x}}$ and express your answer in the simplest form.

SOLUTION

4(a)

$$y = cosec\{sin[ln(x+1)]\}$$

$$\frac{dy}{dx} = -cosec\{sin[ln(x+1)]\}cot\{sin[ln(x+1)]\}\frac{d}{dx}\{sin[ln(x+1)]\}$$

$$\frac{dy}{dx} = -cosec\{sin[ln(x+1)]\}cot\{sin[ln(x+1)]\}\cos[ln(x+1)]\frac{d}{dx}[ln(x+1)]$$

$$\frac{dy}{dx} = -cosec\{sin[ln(x+1)]\}cot\{sin[ln(x+1)]\}\cos[ln(x+1)]\frac{1}{(x+1)}\frac{d}{dx}(x+1)$$

$$\frac{dy}{dx} = -cosec\{sin[ln(x+1)]\}cot\{sin[ln(x+1)]\}\cos[ln(x+1)]\frac{1}{(x+1)}(1)$$

$$\frac{dy}{dx} = \frac{-cosec\{sin[ln(x+1)]\}cot\{sin[ln(x+1)]\}\cos[ln(x+1)]}{(x+1)}$$

4(b)

$$y = \frac{\cos 3x}{e^{2x}}$$

$u = \cos 3x$ $\quad$ $v = e^{2x}$

$u' = -3\sin 3x$ $\quad$ $v' = 2e^{2x}$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{(e^{2x})(-3\sin 3x) - (\cos 3x)(2e^{2x})}{(e^{2x})^2}$$

$$\frac{dy}{dx} = \frac{(e^{2x})[-3\sin 3x - 2\cos 3x]}{(e^{2x})^2}$$

$$\frac{dy}{dx} = \frac{-3\sin 3x - 2\cos 3x}{e^{2x}}$$

$u = -3\sin 3x - 2\cos 3x$ $\quad$ $v = e^{2x}$

$u' = -9\cos 3x + 6sin3x$ $\quad$ $v' = 2e^{2x}$

$$\frac{vu' - uv'}{v^2}$$

$$\frac{d^2y}{dx^2} = \frac{(e^{2x})(-9\cos 3x + 6sin3x) - (-3\sin 3x - 2\cos 3x)(2e^{2x})}{(e^{2x})^2}$$

$$\frac{d^2y}{dx^2} = \frac{(e^{2x})[(-9\cos 3x + 6sin3x) - (2)(-3\sin 3x - 2\cos 3x)]}{(e^{2x})^2}$$

$$\frac{d^2y}{dx^2} = \frac{(e^{2x})[-9\cos 3x + 6sin3x + 6\sin 3x + 4\cos 3x]}{(e^{2x})^2}$$

$$\frac{d^2y}{dx^2} = \frac{-9\cos 3x + 6sin3x + 6\sin 3x + 4\cos 3x}{e^{2x}}$$

$$\frac{d^2y}{dx^2} = \frac{12sin3x - 5\cos 3x}{e^{2x}}$$

5. A cubic polynomial $P(x)$ has remainders 3 and 1 when divided by $(x-1)$ and $(x-2)$, respectively.

(a) Let $Q(x)$ be a linear factor such that $P(x) = (x-1)(x-2)Q(x) + \alpha x + \beta$, where α *and* β are constants. Find the remainder when $P(x)$ is divided by $(x-1)(x-2)$.

(b) Use the values of α *and* β from part (a) to determine $Q(x)$ if the coefficient of x^3 for $P(x)$ is 1 and $P(3) = 7$. Hence, solve for x if $P(x) = 7 - 3x$.

SOLUTION

5(a)

$P(1) = 3$

$P(2) = 1$

$P(x) = (x-1)(x-2)Q(x) + \alpha x + \beta$

$P(1) = (1-1)(1-2)Q(1) + \alpha(1) + \beta = 3$

$\alpha + \beta = 3$ .... (1)

$P(2) = (2-1)(2-2)Q(2) + \alpha(2) + \beta = 1$

$2\alpha + \beta = 1$.............................. (2)

$(1) - (2)$

$\alpha - 2\alpha = 3 - 1$

$-\alpha = 2$

$\alpha = -2$

$\beta = 5$

$P(x) = (x-1)(x-2)Q(x) - 2x + 5$

$$P(x) = D(x)Q(x) + R(x)$$

$$P(x) = \underbrace{(x-1)(x-2)}_{D(x)}Q(x) \underbrace{- 2x + 5}_{R(x)}$$

$\therefore \textit{The remainder when } P(x) \textit{ is divided by } (x-1)(x-2) \textit{ is } -2x + 5$

5(b)

$P(x) = (x-1)(x-2)Q(x) - 2x + 5$

Since the coefficient of x^3 for $P(x)$ is 1 ➔ $Q(x) = (x+c)$

$P(x) = (x-1)(x-2)(x+c) - 2x + 5$

$Given\ that\ P(3) = 7$

$P(3) = (3-1)(3-2)(3+c) - 2(3) + 5 = 7$

$6 + 2c = 8$

$c = 1$

$Q(x) = (x+1)$

$P(x) = (x-1)(x-2)(x+1) - 2x + 5$

$P(x) = 7 - 3x$

$(x-1)(x-2)(x+1) - 2x + 5 = 7 - 3x$

$(x-1)(x-2)(x+1) - 2x + 3x + 5 - 7 = 0$

$(x-1)(x-2)(x+1) + x - 2 = 0$

$(x-1)(x-2)(x+1) + (x-2) = 0$

$(x-2)[(x-1)(x+1) + 1] = 0$

$(x-2)[x^2 + x - x - 1 + 1] = 0$

$(x-2)[x^2] = 0$

$x = 2 \quad or \quad x = 0$

6. (a) State the definition of the continuity of a function at a point. Hence, find the value of d such that

$$f(x) = \begin{cases} e^{3x+d}, & x \leq 0 \\ 3x+5, & x > 0 \end{cases}$$

Is continuous at $x = 0$.

(b) A function f is defined by

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1 \end{cases}$$

Determine the value(s) of k if f is:

(iii) Continuous for all $x \in \mathbb{R}$.
(iv) Differentiable for all $x \in \mathbb{R}$.

SOLUTION

6(a)

Definition of continuity:

A function f is continuous at a point $x = c$ in the domain of f if the following three conditions are satisfied:

i. f(c) is defined
ii. $\lim_{x \to c} f(x)$ exists or finite
iii. $\lim_{x \to c} f(x) = f(c)$

For $f(x)$ to be continuous at x = 0,

(i) $f(0) = e^{3(0)+d} = e^d$

(ii) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3x + 5 = 5$

(iii) $e^d = 5$

$d = ln5$

6(bi)

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} \text{x}^2 - 1 = \lim_{x \to 1^+} k(x-1)$$

$1^2 - 1 = k(1 - 1)$

$0 = 0$

Thus, k can take any real value for the continuity: $k \in \mathbb{R}$

6(bii)

$$f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ k(x-1), & x > 1 \end{cases}$$

$$f'(a) = \lim_{x \to 1^-} \frac{f(x) - f(a)}{x - a}$$

$$f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$f'(1^-) = \lim_{x \to 1^-} \frac{(x^2 - 1) - (1^2 - 1)}{x - 1}$$

$$f'(1^-) = \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1}$$

$$f'(1^-) = \lim_{x \to 1^-} \frac{(x+1)(x-1)}{x - 1}$$

$$f'(1^-) = \lim_{x \to 1^-} (x+1)$$

$$f'(1^-) = 1 + 1$$

$$f'(1^-) = 2$$

$$f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$f'(1^+) = \lim_{x \to 1^+} \frac{(\mathrm{k(x-1)}) - (1^2 - 1)}{x - 1}$$

$$f'(1^+) = \lim_{x \to 1^+} \frac{k(x-1)}{x - 1}$$

$$f'(1^+) = \lim_{x \to 1^+} k$$

$$f'(1^+) = k$$

For f to be differentiable at $x = 1$,

$f'(1^+) = f'(1^-)$

$k = 2$

7. (a) Find the derivative of $f(x) = \frac{1}{x+1}$ by using the first principle.

(b) Use implicit differentiation to find:

(i) $\frac{dy}{dx}$ if $y\ln x = e^{x-y}$.

(ii) the value of $\frac{dy}{dx}$ if $\frac{1}{y} - \frac{1}{x} = 3$ when $x = \frac{1}{2}$.

SOLUTION

7(a)

$$f(x) = \frac{1}{x+1}$$

$$f(x+h) = \frac{1}{(x+h)+1}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(x+h)+1} - \frac{1}{x+1}\right)$$

First principle

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{h}\big(f(x+h) - f(x)\big)$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1(x+1) - 1(x+h+1)}{(x+h+1)(x+1)}\right)$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{h}\left(\frac{x+1-x-h-1}{(x+h+1)(x+1)}\right)$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{(x+h+1)(x+1)}\right)$$

$$\frac{dy}{dx} = \lim_{h\to 0}\left(\frac{-1}{(x+h+1)(x+1)}\right)$$

$$\frac{dy}{dx} = \frac{-1}{(x+0+1)(x+1)}$$

$$\frac{dy}{dx} = \frac{-1}{(x+1)^2}$$

7(bi)

$$y\ln x = e^{x-y}$$

$$y\left(\frac{1}{x}\right) + \ln x\left(\frac{dy}{dx}\right) = e^{x-y}\frac{d}{dx}(x-y)$$

$$\frac{y}{x} + \ln x\left(\frac{dy}{dx}\right) = e^{x-y}\left[1 - \frac{dy}{dx}\right]$$

$$\frac{y}{x} + lnx\left(\frac{dy}{dx}\right) = e^{x-y} - e^{x-y}\frac{dy}{dx}$$

$$lnx\left(\frac{dy}{dx}\right) + e^{x-y}\frac{dy}{dx} = e^{x-y} - \frac{y}{x}$$

$$\left(\frac{dy}{dx}\right)[lnx + e^{x-y}] = e^{x-y} - \frac{y}{x}$$

$$\left(\frac{dy}{dx}\right) = \frac{e^{x-y} - \frac{y}{x}}{lnx + e^{x-y}}$$

7(bii)

$$\frac{1}{y} - \frac{1}{\mathrm{x}} = 3$$

$$when\ x = \frac{1}{2}$$

$$\frac{1}{y} - \frac{1}{\left(\frac{1}{2}\right)} = 3$$

$$\frac{1}{y} - 2 = 3$$

$$\frac{1}{y} = 3 + 2$$

$$\frac{1}{y} = 5$$

$$y = \frac{1}{5}$$

$$\frac{1}{y} - \frac{1}{\mathrm{x}} = 3$$

$$y^{-1} - x^{-1} = 3$$

$$(-1)y^{-2}\frac{dy}{dx} - (-1)x^{-2} = 0$$

$$\frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{x^2} = 0$$

$$\frac{-1}{y^2}\frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}.\frac{y^2}{-1}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

$$When\ x = \frac{1}{2}, \qquad y = \frac{1}{5}$$

$$\frac{dy}{dx} = \frac{\left(\frac{1}{5}\right)^2}{\left(\frac{1}{2}\right)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{25}}{\frac{1}{4}}$$

$$\frac{dy}{dx} = \frac{1}{25}.\frac{4}{1}$$

$$\frac{dy}{dx} = \frac{4}{25}$$

8. A curve is defined by parametric equations

$$x = \ln(1+t), \qquad y = e^{t^2} \qquad \text{for } t > -1.$$

(c) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

(d) Show that the curve has only one relative extremum at (0,1) and determine the nature of the point.

SOLUTION

8(a)

$x = \ln(1+t),$ $\qquad$ $y = e^{t^2}$

$\frac{dx}{dt} = \frac{1}{1+t}\frac{d}{dt}(1+t)$ $\qquad$ $\frac{dy}{dt} = e^{t^2}\frac{d}{dt}(t^2)$

$\frac{dx}{dt} = \frac{1}{1+t}(1)$ $\qquad$ $\frac{dy}{dt} = e^{t^2}(2t)$

$\frac{dx}{dt} = \frac{1}{1+t}$ $\qquad$ $\frac{dy}{dt} = 2te^{t^2}$

$\frac{dt}{dx} = 1+t$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx} = \left(2te^{t^2}\right).(1+t)$$

$$\frac{dy}{dx} = 2te^{t^2}(1+t)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right].\left[\frac{dt}{dx}\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right].\left[\frac{dt}{dx}\right]$$

$$\frac{d^2y}{dx^2} = \left\{\frac{d}{dt}\left[2te^{t^2}(1+t)\right]\right\}.[1+t]$$

$u = 2te^{t^2}$ $\qquad$ $v = 1+t$

$u' = (2t)\left(2te^{t^2}\right) + \left(e^{t^2}\right)(2)$ $\qquad$ $v' = 1$

$u' = 4t^2e^{t^2} + 2e^{t^2}$

$$\frac{d^2y}{dx^2} = \left\{\left(2te^{t^2}\right)(1) + (1+t)\left(4t^2e^{t^2} + 2e^{t^2}\right)\right\}.[1+t]$$

$$\frac{d^2y}{dx^2} = \{2te^{t^2} + 4t^2e^{t^2} + 2e^{t^2} + 4t^3e^{t^2} + 2te^{t^2}\}.[1+t]$$

$$\frac{d^2y}{dx^2} = \{2e^{t^2} + 4te^{t^2} + 4t^2e^{t^2} + 4t^3e^{t^2}\}.[1+t]$$

$$\frac{d^2y}{dx^2} = \{2e^{t^2}(1 + 2t + 2t^2 + 2t^3)\}.[1+t]$$

8(b)

For extremum point(s), let $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 2te^{t^2}(1+t)$$

$$2te^{t^2}(1+t) = 0$$

$t = 0$ or $t = -1$

Since given that $t > -1$, the curve has only one extremum point when $t = 0$.

Whent $t = 0$,

$x = \ln(1+t),$	$y = e^{t^2}$
$x = \ln(1+0),$	$y = e^{0^2}$
$x = \ln(1),$	$y = e^0$
$x = 0,$	$y = 1$

⇨ (0, 1) is the extremum point.

At $t = 0$, the second derivative test gives

$$\frac{d^2y}{dx^2} = \{2e^{0^2}(1 + 2(0) + 2(0^2) + 2(0^3))\}.[1+(0)]$$

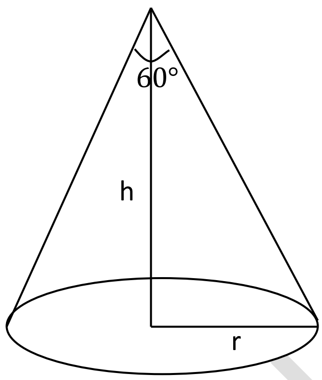
$$\frac{d^2y}{dx^2} = \{2(1)(1+0+0+0)\}.[1+(0)]$$

$$\frac{d^2y}{dx^2} = 2 > 0$$

Thus, the extremum point (0,1) is a relative minimum point.

9. (a) A cylindrical container of volume $128\pi\ m^3$ is to be constructed with the same material for the top, bottom and lateral side. Find the dimensions of the container that will minimise the amount of the material needed.

(b) Gravel is poured onto a flat ground at the rate of $\frac{3}{20}\ m^3$ per minute to form a conical-shaped pile with vertex angle 60° as shown in the diagram below.



Compute the rate of chang of the height of the conical pile at the instant t = 10 minutes.

SOLUTION

$V = \pi r^2 h = 128\pi$ ➔ To minimise the ***surface area***.

$h = \frac{128\pi}{\pi r^2}$

$h = \frac{128}{r^2}$

To minimise the ***surface area***

$$S = 2\pi r^2 + 2\pi rh$$

$$S = 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right)$$

$$S = 2\pi r^2 + \frac{256\pi}{r}$$

$$S = 2\pi r^2 + 256\pi r^{-1}$$

$$\frac{dS}{dr} = 4\pi r - 256\pi r^{-2}$$

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

$$Let\,\frac{dS}{dr} = 0$$

$$4\pi r - \frac{256\pi}{r^2} = 0$$

$$\frac{256\pi}{r^2} = 4\pi r$$

$$4\pi r^3 = 256\pi$$

$$r^3 = \frac{256\pi}{4\pi}$$

$$r^3 = \frac{256}{4}$$

$$r^3 = 64$$

$$r = 4$$

When $r = 4$,

$$h = \frac{128}{r^2}$$

$$h = \frac{128}{4^2}$$

$$h = 8$$

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$$

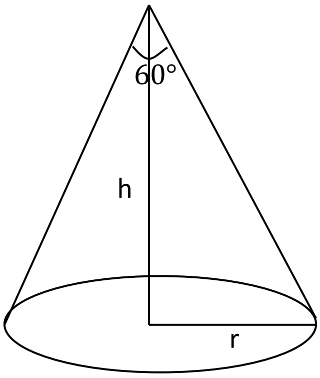
$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

$$When\, r = 4$$

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{4^3} = 12\pi > 0$$

$$h = 8\ and\ r = 4\ give\ minimum\ total\ surface\ area\ by\ the\ second\ derivative\ test$$

9(b)



$\frac{dV}{dt} = \frac{3}{20}\ m^3$ Find $\frac{dh}{dt}$ when $t\ =\ 10\ minutes$

$$\tan 30° = \frac{r}{h}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{h}$$

$$\sqrt{3}r = h$$

$$r = \frac{h}{\sqrt{3}}$$

$$\frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h^2}{3}\right)h$$

$$V = \frac{\pi}{9}h^3$$

$$\frac{dV}{dh} = \frac{\pi}{3}h^2$$

$$\frac{dV}{dh} = \frac{\pi h^2}{3}$$

$$\frac{dh}{dV} = \frac{3}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{dh}{dV}.\frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \cdot \frac{3}{20}$$

$$\frac{dh}{dt} = \frac{9}{20\pi h^2}$$

$When\ t = 10,$

$$\frac{dV}{dt} = \frac{3}{20} \rightarrow V = \frac{3}{20}t$$

$$V = \frac{3}{20}(10) = \frac{3}{2}$$

$When\ V = \frac{3}{2}$

$$\frac{3}{2} = \frac{\pi}{9}h^3$$

$$h^3 = \frac{9}{\pi} \cdot \frac{3}{2}$$

$$h^3 = \frac{27}{2\pi}$$

$$h = \left(\frac{27}{2\pi}\right)^{\frac{1}{3}}$$

$$\frac{dh}{dt} = \frac{9}{20\pi\left(\frac{27}{2\pi}\right)^{\frac{2}{3}}}$$

$$\frac{dh}{dt} = \frac{9}{20\pi}\left(\frac{2\pi}{27}\right)^{\frac{2}{3}} = 0.0542$$

10. (a) Show that $\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta} = \cot\left[\frac{\beta-\alpha}{2}\right]$.

(b) Use trigonometric identities to verify that

(i) $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

Hence, solve the equation $3\sin\theta + \cos\theta = 2$ for $0° \le \theta \le 180°$. Give your answers correct to three decimal places.

SOLUTION

10(a)

$$\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{-\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$= -cot\left(\frac{\alpha-\beta}{2}\right)$$

$$= cot\left(\frac{\beta-\alpha}{2}\right)$$

10(*bi*)

$$\sin\theta = 2sin\frac{\theta}{2}cos\frac{\theta}{2}$$

$$= 2sin\frac{\theta}{2}cos\frac{\theta}{2}\left[\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right]$$

$$= \frac{2sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= 2tan\frac{\theta}{2}\cos^2\frac{\theta}{2}$$

$$= 2tan\frac{\theta}{2}\frac{1}{\frac{1}{\cos^2\frac{\theta}{2}}}$$

$$= 2tan\frac{\theta}{2}\frac{1}{\sec^2\frac{\theta}{2}}$$

$$= \frac{2tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$= \frac{2tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

10(*bii*)

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

$$= \left(1 - 2\sin^2\frac{\theta}{2}\right).\left[\frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right]$$

$$= \frac{\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}$$

$$= \left(\frac{1}{\cos^2\frac{\theta}{2}} - \frac{2\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right).\cos^2\frac{\theta}{2}$$

$$= \left(\sec^2\frac{\theta}{2} - 2\tan^2\frac{\theta}{2}\right).\frac{1}{\frac{1}{\cos^2\frac{\theta}{2}}}$$

$$= \left(\sec^2\frac{\theta}{2} - 2\tan^2\frac{\theta}{2}\right).\frac{1}{\sec^2\frac{\theta}{2}}$$

$$= \frac{\sec^2\frac{\theta}{2} - 2\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}}$$

$$= \frac{\left(1+\tan^2\frac{\theta}{2}\right) - 2\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$= \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}$$

10(b)

$$3 \sin \theta + \cos \theta = 2$$

$$3\left(\frac{2tan\frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}\right) + \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = 2$$

Let $t = tan\frac{\theta}{2}$

$$3\left(\frac{2t}{1 + t^2}\right) + \frac{1 - t^2}{1 + t^2} = 2$$

$$\frac{6t + 1 - t^2}{1 + t^2} = 2$$

$$6t + 1 - t^2 = 2(1 + t^2)$$

$$6t + 1 - t^2 = 2 + 2t^2$$

$$6t + 1 - t^2 = 2 + 2t^2$$

$$2 + 2t^2 + t^2 - 6t - 1 = 0$$

$$3t^2 - 6t + 1 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(1)}}{2(3)}$$

$$t = \frac{6 \pm \sqrt{36 - 12}}{6}$$

$t = 0.1835$ or $t = 1.8165$

$tan\frac{\theta}{2} = 0.1835$ $\qquad$ $tan\frac{\theta}{2} = 1.8165$

$\frac{\theta}{2} = \tan^{-1} 0.1835$ $\qquad$ $\frac{\theta}{2} = \tan^{-1} 1.8165$

$\frac{\theta}{2} = 10.398°$ $\qquad$ $\frac{\theta}{2} = 61.167°$

$\theta = 20.796°$ $\qquad$ $\theta = 122.334°$