Chapter 4: Matrices and Systems of Linear Equations

4.4 System of Linear Equations with Three Variables

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Learning Outcomes

(a) Write a system of linear equations in the form of $\boldsymbol{AX = B}$

**Up to 3x3 matrices.*

*** Apply to some practical problems.*

(b) Solve the unique solution of $\boldsymbol{AX = B}$ using:

(i) Inverse Matrix;

(ii) Elimination Method,

** Introduce Gauss-Jordan*

Matrix Equation

The system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Can be written in the form $AX = B$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Matrix of coefficients

Matrix of variables

Matrix of constants

Example

(1) Write the following system of linear equations in the form $AX = B$.

$$x + 2y - 6z = 11$$
$$2x - y + 2z = 3$$
$$3x - 2y - 2z = 9$$

Solution:

$$\begin{pmatrix} 1 & 2 & -6 \\ 2 & -1 & 2 \\ 3 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \\ 9 \end{pmatrix}$$

Bloom: Understanding

Methods of solving

Methods of solving $AX = B$

1. Using inverse matrix

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$ Multiplying each side on the left by A^{-1}

$$IX = A^{-1}B$$ $A^{-1}A = I$

$$X = A^{-1}B$$ $IX = X$

Methods of solving

Methods of solving $AX = B$

2. Gauss-Jordan elimination method

Step 1: Write in the form of $AX = B$.

Step 2: Form an augmented matrix $(A|B)$.

Step 3: Use elementary row operations (ERO) to reduce the augmented matrix $(A|B)$ to a reduced augmented form $(I|C)$.

Example

(1) By using the inverse matrix, solve the following system of linear equations

$$\begin{aligned} x + y + 2z &= 3 \\ -2x + 3y + 4z &= -3 \\ 5x - 4y - z &= 13 \end{aligned}$$

(2) By using the Gauss-Jordan elimination method, solve the following system of linear equations.

$$\begin{aligned} x + y &= 0 \\ 2x + 3y + 3z &= 1 \\ -x + y + z &= 1 \end{aligned}$$

Bloom: Understanding

Solution

(1) By using inverse matrix

$$\underbrace{\begin{pmatrix} 1 & 1 & 2 \\ -2 & 3 & 4 \\ 5 & -4 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 3 \\ -3 \\ 13 \end{pmatrix}}_{B}$$

Writing the system of equations in the form $AX = B$

$$|A| = (+)(1)\begin{vmatrix} 3 & 4 \\ -4 & -1 \end{vmatrix} (-)(1)\begin{vmatrix} -2 & 4 \\ 5 & -1 \end{vmatrix} (+)(2)\begin{vmatrix} -2 & 3 \\ 5 & -4 \end{vmatrix}$$

$$= (-3 + 16) - (2 - 20) + 2(8 - 15)$$

$$= 17$$

Expanding about the first row.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C_A = \begin{bmatrix} +\begin{vmatrix} 3 & 4 \\ -4 & -1 \end{vmatrix} & -\begin{vmatrix} -2 & 4 \\ 5 & -1 \end{vmatrix} & +\begin{vmatrix} -2 & 3 \\ 5 & -4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ -4 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & -4 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 13 & 18 & -7 \\ -7 & -11 & 9 \\ -2 & -8 & 5 \end{bmatrix}$$

Bloom: Understanding

Solution

(1) By using inverse matrix (Continue...)

$$Adj\ A = C_A{}^T = \begin{bmatrix} 13 & -7 & -2 \\ 18 & -11 & -8 \\ -7 & 9 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 13 & -7 & -2 \\ 18 & -11 & -8 \\ -7 & 9 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{17}\begin{bmatrix} 13 & -7 & -2 \\ 18 & -11 & -8 \\ -7 & 9 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 13 \end{bmatrix}$$

$$X = \frac{1}{17}\begin{pmatrix} 39 + 21 - 26 \\ 54 + 33 - 104 \\ -21 - 27 + 65 \end{pmatrix}$$

$$= \frac{1}{17}\begin{pmatrix} 34 \\ -17 \\ 17 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore x = 2, \qquad y = -1, \qquad z = 1$$

Solution

(2) By using Gauss-Jordan elimination method

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 3 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Writing the system of equations in the form $AX = B$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right)$$

Forming the augmented matrix $(A|B)$

$$R_2{}^* = -2R_1 + R_2$$

$$R_3{}^* = R_1 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 1 & 1 \end{array}\right)$$

Changing the elements in the 2nd and 3rd rows of the 1st column to 0.

Solution

(2) By using Gauss-Jordan elimination method (Continue…)

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 1 & 1 \end{array}\right)$$

$$R_3{}^* = -2R_2 + R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -1 \end{array}\right)$$

Changing the element in the 3rd row of the 2nd column to 0.

$$R_3{}^* = -\frac{1}{5}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right)$$

Changing the element in the 3rd row of the 3rd column to 0.

Solution

(2) By using Gauss-Jordan elimination method (Continue…)

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right)$$

$$R_2{}^* = -3R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right)$$

Changing the element in the 2nd row of the 3rd column to 0.

Solution

(2) By using Gauss-Jordan elimination method (Continue…)

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right)$$

$$R_1^* = -R_2 + R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{5} \end{array}\right)$$

Changing the element in the 1st row of the 2nd column to 0.

$$\therefore x = -\frac{2}{5}, \quad y = \frac{2}{5}, \quad z = \frac{1}{5}$$

Self-check

(1) By using the inverse matrix, solve the following system of linear equations

$$\begin{aligned} 4x + y + 2z &= 13 \\ -2x + 3y + z &= -9 \\ -x + 3y + z &= -7 \end{aligned}$$

(2) By using the Gauss-Jordan elimination method, solve the following system of linear equations.

$$\begin{aligned} 5x - 3y + 2z &= 13 \\ -2x + y + 3z &= -1 \\ 2x - y + 2z &= 6 \end{aligned}$$

Answer Self-check

(1) $x = 2, \quad y = -3, \quad z = 4$

(2) $x = 1, \quad y = -2, \quad z = 1$

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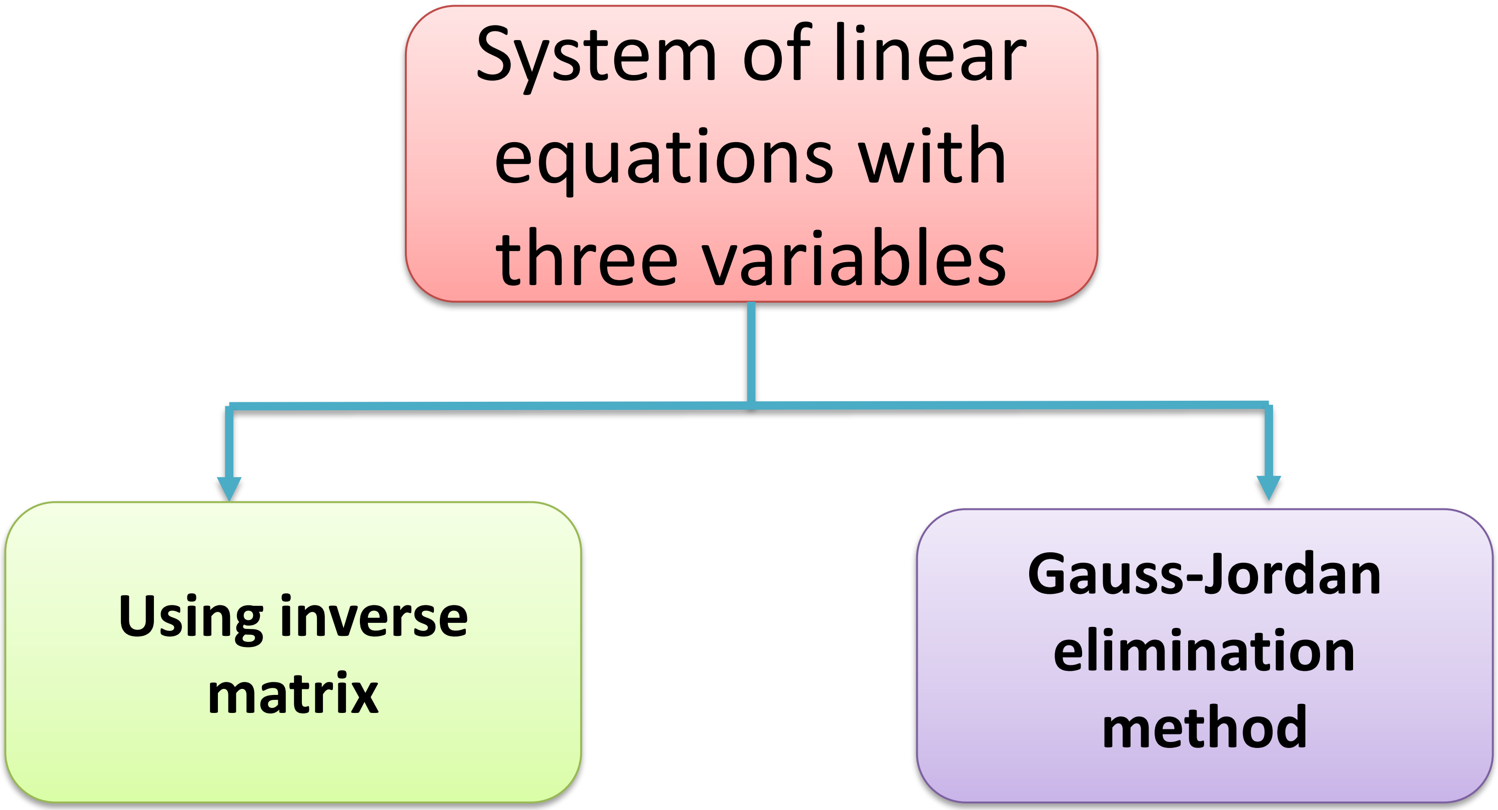
Bloom: Applying

Summary



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Bloom: Remembering

Key Terms

- Inverse matrix method
- Gauss-Jordan elimination method
- Augmented matrix
- Matrix of coefficients
- Matrix of variables
- Matrix of constants