Chapter 4: Matrices and Systems of Linear Equations

4.3 Inverse of a Matrix

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Learning Outcomes

(a) Compute the inverse of a non singular matrix using :

(i) Adjoin Matrix

(ii) Elementary row operations.

**Up to 3x3 matrices.*

Inverse Matrices

The inverse matrix can be obtained by

(i) Formula

$$A^{-1} = \frac{1}{|A|} Adj\ A \quad \text{where} \quad Adj\ (A) = \left(C_{ij}\right)^T$$

Step 1: Find determinant.
Step 2: Find cofactor of each element.
Step 3: Find adjoint matrix.
Step 4: Apply the formula of inverse matrix.

(ii) Elementary row operation (ERO)

Step 1: Form the augmented matrix $(A|I)$.
Step 2: Use elementary row operation to transform $(A|I)$ into $\left(I|A^{-1}\right)$.

Inverse Matrices

Elementary Row Operations (ERO)

- ERO are manipulations that can be performed on the row of a matrix.
- There are three basic elementary row operations.

(i) Interchange any two rows $\left(R_i{}^* = R_j\right)$

Example: $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1{}^* = R_2} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(ii) Multiply row R_i by a non-zero scalar, k $\left(R_i{}^* = kR_i\right)$

Example: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2{}^* = \frac{1}{3}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

(iii) Add k times row R_i to row R_j $\left(R_j{}^* = kR_i + R_j\right)$

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2{}^* = (-3)R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

Example (Formula)

(1) Find $\boldsymbol{A^{-1}}$; given $\boldsymbol{A} = \begin{pmatrix} \mathbf{6} & \mathbf{8} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$ by using adjont matrix.

(2) Find the inverse of matrix $\boldsymbol{B} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$ by using adjoint matrix.

Solution (Formula)

(1) $|A| = (6)(2) - (-1)(8)$
$= 12 + 8$
$= 20$

$$C_A = \begin{bmatrix} +a_{22} & -a_{21} \\ -a_{12} & +a_{11} \end{bmatrix}$$

$$C_A = \begin{bmatrix} 2 & 1 \\ -8 & 6 \end{bmatrix}$$

$$Adj\ A = {C_A}^T = \begin{bmatrix} 2 & 1 \\ -8 & 6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -8 \\ 1 & 6 \end{bmatrix}$$

$$A = \begin{pmatrix} 6 & 8 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

$$C_{11} = (+)a_{22}$$

Deleting 1st row 1st column, so getting a_{22}

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Solution (Formula)

(1) Continue…

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{20}\begin{bmatrix} 2 & -8 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} & -\frac{2}{5} \\ \frac{1}{20} & \frac{3}{10} \end{bmatrix}$$

Solution (Formula)

(2) **Firstly, find the determinant of the matrix.**

Choose 1st row

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$|B| = (+)(1)\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} (-)(1)\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} (+)(-1)\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 3 - 1 - 1$$

$$= 1$$

Then, find the cofactor of each element.

$$C_B = \begin{bmatrix} +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

Solution (Formula)

(2) Continue…

$$C_B = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$Adj\ B = {C_B}^T = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} Adj\ B$$

$$= \frac{1}{(1)} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Self-check

(1) Find A^{-1} ; given $A = \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$ by using adjont matrix.

(2) Find the inverse of matrix $B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

by using adjoint matrix.

Answer Self-check

(1) $A^{-1} = \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix}$

(2) $B^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

Bloom: Applying

Example (ERO)

(1) Find A^{-1} ; given $A = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$. Use elementary row operations.

(2) Find the inverse of matrix $B = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 5 & 9 \\ 1 & 1 & -2 \end{pmatrix}$

by using elementary row operations.

Solution (ERO)

(1) ***Step 1: Form the augmented matrix***

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & -4 & 0 & 1 \end{array}\right)$$

$(A|I)$

Step 2: Use ERO operation

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & -4 & 0 & 1 \end{array}\right)$$

$$R_2{}^* = (-2)R_1 + R_2$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -10 & -2 & 1 \end{array}\right)$$

Solution (ERO)

(1) Continue…

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -10 & -2 & 1 \end{array}\right)$$

$$R_2{}^* = \frac{1}{-10} R_2$$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{1}{10} \end{array}\right) \xrightarrow{R_1{}^* = (-3)R_2 + R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{3}{10} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{10} \end{array}\right)$$

In the form of $(I|A^{-1})$

$$\therefore A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{3}{10} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

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Solution (ERO)

(2) ***Step 1: Form the augmented matrix***

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 3 & 5 & 9 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$(B|I)$

Step 2: Use ERO operation

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 3 & 5 & 9 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$R_2^{*} = (-3)R_1 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Solution (ERO)

(2) Continue…

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow \quad {R_3}^* = (-1)R_1 + R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & -6 & -3 & 1 & 0 \\ 0 & -1 & -7 & -1 & 0 & 1 \end{array}\right)$$

$$\downarrow \quad {R_2}^* = (-1)R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & -1 & -7 & -1 & 0 & 1 \end{array}\right)$$

Solution (ERO)

(2) Continue…

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & -1 & -7 & -1 & 0 & 1 \end{array}\right)$$

$$\downarrow \; R_3{}^* = R_2 + R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right)$$

$$\downarrow \; R_3{}^* = (-1)R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

Solution (ERO)

(2) Continue…

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 6 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

$$\downarrow \; R_2{}^* = (-6)R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 15 & -7 & 6 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

$$\downarrow \; R_1{}^* = (-5)R_3 + R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 11 & -5 & 5 \\ 0 & 1 & 0 & 15 & -7 & 6 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

Solution (ERO)

(2) Continue…

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 11 & -5 & 5 \\ 0 & 1 & 0 & 15 & -7 & 6 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

$$\downarrow \quad {R_1}^* = (-2)R_2 + R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & 9 & -7 \\ 0 & 1 & 0 & 15 & -7 & 6 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

In the form of $\left(I \middle| B^{-1}\right)$

$$\therefore B^{-1} = \begin{bmatrix} -19 & 9 & -7 \\ 15 & -7 & 6 \\ -2 & 1 & -1 \end{bmatrix}$$

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Bloom: Understanding

Self-check

(1) Find $\boldsymbol{A^{-1}}$; given $\boldsymbol{A = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}}$. Use elementary row operations.

(2) Find the inverse of matrix $\boldsymbol{B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}}$

by using elementary row operations.

Answer Self-check

(1) $A^{-1} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

(2) $B^{-1} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Bloom: Applying

Summary

Inverse Matrices

- **Formula**

  $$A^{-1} = \frac{1}{|A|} Adj\ A$$

- **Elementary Row Operations (ERO)**

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Bloom: Remembering

Key Terms

- Inverse matrix
- Elementary row operations
- Adjoint matrix
- Augmented matrix
- Determinant
- Cofactor