# Chapter 4: Matrices and Systems of Linear Equations

## 4.2 Determinant of Matrices

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# Learning Outcomes

(a) Find the minors and cofactors of a matrix

**Up to 3x3 matrices.*

(b) Find the determinant of a matrix.

**Use basic properties of determinant.*

# Minors

## Minors, $M_{ij}$

**Minors of an element of an $3 \times 3$ matrix is determinant by deleting the row and the column containing the element and then finding the determinant of the resulting $2 \times 2$ matrix.**

***Example:***

**If** $A = \begin{pmatrix} 3 & -1 & 4 \\ -2 & 5 & 2 \\ 1 & 4 & -3 \end{pmatrix}$, **find** $M_{11}$ **and** $M_{23}$ .

# Minors

***Solution:***

$$A = \begin{pmatrix} 3 & -1 & 4 \\ -2 & 5 & 2 \\ 1 & 4 & -3 \end{pmatrix}$$

Deleting the 1st row and the 1st column.

$$M_{11} = \begin{vmatrix} 5 & 2 \\ 4 & -3 \end{vmatrix}$$

$$= (5)(-3) - (4)(2)$$

$$= -15 - 8$$

$$= -23$$

Applying the formula

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

# Minors

***Solution:***

$$A = \begin{pmatrix} 3 & -1 & 4 \\ -2 & 5 & 2 \\ 1 & 4 & -3 \end{pmatrix}$$

**Deleting the 2nd row and the 3rd column.**

$$M_{23} = \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= (3)(4) - (1)(-1)$$

$$= 12 + 1$$

$$= 13$$

**Applying the formula**

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

# Cofactors

## Cofactors

If $A$ is a square matrix, then the cofactor, denoted by $c_{ij}$, of the element $a_{ij}$ is given by $c_{ij} = (-1)^{i+j} M_{ij}$.

***Learning Tips:***

Instead of calculating $(-1)^{i+j}$ for each of minor, the following sign convention can be used to find cofactors of matrices.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

# Cofactors

**Example:**

If $A = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 3 & -1 \\ 2 & 1 & -3 \end{pmatrix}$, find

(a) $c_{12}$

(b) $c_{33}$

**Solution:**

(a) $A = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 3 & -1 \\ 2 & 1 & -3 \end{pmatrix}$

Deleting the 1st row and the 2nd column.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$c_{12}$ the sign is negative

$$c_{12} = (-)M_{12}$$

$$= (-)\begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix}$$

$$= -8$$

# Cofactors

***Solution:***

(b)
$$A = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 3 & -1 \\ 2 & 1 & -3 \end{pmatrix}$$

Deleting the 3rd row and the 3rd column.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$c_{33}$ the sign is positive

$$c_{33} = (+)M_{33}$$
$$= (+)\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix}$$
$$= 12 - (-2)$$
$$= 14$$

# Determinant of matrices

## Determinant of a $2 \times 2$ matrix

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $|A| = a_{11}a_{22} - a_{21}a_{12}$.

***Example:***

If $A = \begin{pmatrix} 3 & -2 \\ -4 & -1 \end{pmatrix}$, find $|A|$.

***Solution:***

$$|A| = \begin{vmatrix} 3 & -2 \\ -4 & -1 \end{vmatrix}$$

$$= (3)(-1) - (-4)(-2)$$

$$= -11$$

**Applying the formula**

$$|A| = a_{11}a_{22} - a_{21}a_{12}$$

# Determinant of matrices

## Determinant of a $3 \times 3$ matrix

If $A = \begin{pmatrix} 4 & 1 & 0 \\ -2 & 3 & -1 \\ 2 & 1 & -3 \end{pmatrix}$, then $|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$

***Learning Tips:***

**Step 1: Fix any row or column to find determinant.**
**Step 2: Label the sign according to the sign convention.**
**Step 3: Draw a bracket and modulus after each sign.**
**Step 4: Fill in the value of elements in bracket and the elements of minor in modulus then calculate to get determinant.**

# Determinant of matrices

***Example:***

**Evaluate** $\begin{vmatrix} 1 & -3 & 2 \\ 3 & 4 & -2 \\ -2 & 0 & 0 \end{vmatrix}$.

***Solution:***

*Hint:* **Choose row or column contains the most zeroes to make computation easier.**

$$\begin{vmatrix} 1 & -3 & 2 \\ 3 & 4 & -2 \\ -2 & 0 & 0 \end{vmatrix} = +(\quad)\begin{vmatrix} & \\ & \end{vmatrix} - (\quad)\begin{vmatrix} & \\ & \end{vmatrix} + (\quad)\begin{vmatrix} & \\ & \end{vmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

We expand about the 3rd row with most zeroes. The sign is + - +

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*Bloom: Remembering*

# Determinant of matrices

***Solution: (Continue…)***

$$\begin{vmatrix} 1 & -3 & 2 \\ 3 & 4 & -2 \\ -2 & 0 & 0 \end{vmatrix}$$

$$= +(-2)\begin{vmatrix} -3 & 2 \\ 4 & -2 \end{vmatrix} - (0)\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} + (0)\begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix}$$

Fill in the values of elements in bracket and modulus.

$$= (-2)(6-8) - 0 + 0$$

$$= 4$$

# Properties of determinant

1. If a square matrix B is obtained from a square matrix A by multiplying each element of any row or any column of matrix A by a constant $k$ , then $|B| = k|A|$ .

**Example:**

If $A = \begin{pmatrix} 2 & 4 & 0 \\ 1 & -3 & 1 \\ 5 & 2 & 0 \end{pmatrix}$ and $|A| = 16$.

$$|B| = \begin{vmatrix} 2 & 4 & 0 \\ 3 & -9 & 3 \\ 5 & 2 & 0 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 0 \\ 1 & -3 & 1 \\ 5 & 2 & 0 \end{vmatrix} = 3(16) = 48$$

# Properties of determinant

2. If all the elements in a row or a column of a square matrix A are zeroes, then $|A| = 0$

***Example:***

$$A = \begin{pmatrix} 2 & 5 & 2 \\ 0 & 0 & 0 \\ 5 & 2 & 1 \end{pmatrix} \quad \therefore \ |A| = 0$$

All elements in 2nd row are zeroes.

$$B = \begin{pmatrix} 6 & 8 & 0 \\ 2 & 7 & 0 \\ 5 & 1 & 0 \end{pmatrix} \quad \therefore \ |B| = 0$$

All elements in 3rd column are zeroes.

# Properties of determinant

3. If any two rows or two columns of a square matrix A are identical, then $|A| = 0$.

***Example:***

$$A = \begin{pmatrix} 2 & 5 & 8 \\ 5 & 3 & 9 \\ 2 & 5 & 8 \end{pmatrix} \quad \therefore\ |A| = 0$$

The 1st and 3rd rows are identical.

$$B = \begin{pmatrix} 4 & 4 & 3 \\ 2 & 2 & 7 \\ 6 & 6 & 9 \end{pmatrix} \quad \therefore\ |B| = 0$$

The 1st and 2nd columns are identical.

# Properties of determinant

**4. If A is a square matrix, then $|A| = |A^T|$.**

***Example:***

If $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ -2 & -1 & 1 \end{pmatrix}$ and $|A| = 12$.

$$A^T = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 3 & -1 \\ 2 & -1 & 1 \end{pmatrix} \quad \therefore |A^T| = |A| = 12$$

# Properties of determinant

5. **If a square matrix B is obtained from a square matrix A by interchanging any two rows or any two columns, then $|B| = -|A|$.**

***Example:***

Given $\begin{vmatrix} a & k & p \\ b & l & q \\ c & m & r \end{vmatrix} = 10$.

$$\begin{vmatrix} c & m & r \\ b & l & q \\ a & k & p \end{vmatrix} = -\begin{vmatrix} a & k & p \\ b & l & q \\ c & m & r \end{vmatrix} = -10$$

Interchanging the 1st row with the 3rd row.

# Properties of determinant

**6. If A is an upper triangular or a lower triangular matrix, then $|A|$ can be obtained by multiplying the elements on the leading diagonal.**

***Example:***

$$A = \begin{pmatrix} 1 & 4 & -3 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

**Upper triangular matrix.**

$$|A| = (1)(-2)(-1)$$

$$= 2$$

**Multiplying the elements on the leading diagonal.**

# Properties of determinant

**7. If A and B are two square matrices of the same order, then $|AB| = |A| \times |B|$.**

***Example:***

**Given that** $|A| = \begin{vmatrix} 3 & -2 & 4 \\ 1 & 0 & 0 \\ 5 & 3 & -1 \end{vmatrix} = 10$ **and**

$$|B| = \begin{vmatrix} 1 & 5 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 2 \end{vmatrix} = -2.$$

$$\therefore |AB| = |A| \times |B| = (10)(-2) = -20$$

# Example

(1) Find the determinant of $A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$.

(2) Evaluate $|A|$ if $A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 3 & 2 \\ -2 & -4 & 5 \end{bmatrix}$.

(3) Given $A = \begin{bmatrix} 2 & y & -1 \\ y & 2 & 1 \\ -3 & -1 & 1 \end{bmatrix}$. Find $y$ if $|A| = 0$ .



*Bloom: Understanding*

# Solution

(1) $|A| = (3)(-1) - (2)(4)$

$= -11$

(2) **Expand about the 1st row.**

$$|A| = +(3)\begin{vmatrix} 3 & 2 \\ -4 & 5 \end{vmatrix} - (5)\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} + (4)\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$$

$$= 3(15+8) - 5(5+4) + 4(-4+6)$$

$$= 32$$

# Solution (Continue…)

(3) **Expand about the 1st row.**

$$|A| = +(2)\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - (y)\begin{vmatrix} y & 1 \\ -3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} y & 2 \\ -3 & -1 \end{vmatrix} = 0$$

$$2[(2) - (-1)] - y[y + 3] - [-y + 6] = 0$$

$$y^2 + 2y = 0$$

$$y(y + 2) = 0$$

$$\therefore y = 0 \text{ or } y = -2$$

# Self-check

(1) Find the determinant of $\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$.

(2) Evaluate $|\boldsymbol{A}|$ if $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix}$.

(3) Given $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & \boldsymbol{k} \\ -1 & \boldsymbol{k} & 1 \end{bmatrix}$. Find $\boldsymbol{k}$ if $|\boldsymbol{A}| = \boldsymbol{0}$.

# Answer Self-check

**(1) 10**

**(2) -41**

**(3)** $\boldsymbol{k = -6 \ or \ k = -1}$

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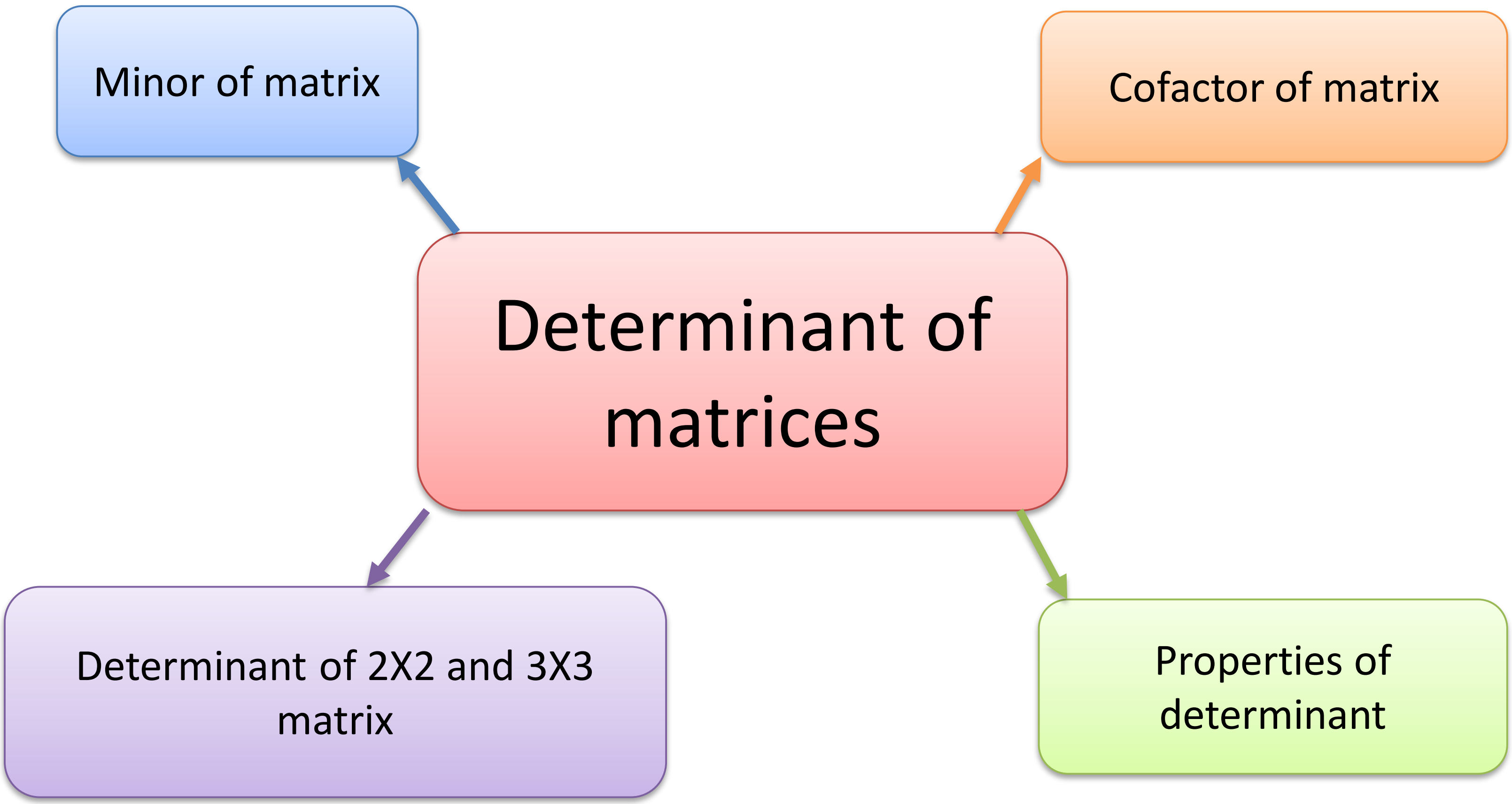
*Bloom: Applying*

# Summary

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*Bloom: Remembering*

# Key Terms

- Minor
- Cofactor
- Determinant