Chapter 4: Matrices and Systems of Linear Equations

4.1 Matrices

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Learning Outcomes

(a) Identify the different types of matrices.

(b) Perform operations on matrices.

(c) Find the transpose of a matrix.

Bloom: Understanding

Matrices

3 is the **element** in the 1st row 1st column

$$\begin{pmatrix} 3 & 7 & 8 \\ 5 & 2 & 1 \end{pmatrix}$$

Row 1

Row 2

Column 1

Column 2

Column 3

Matrix 2×3 since it has 2 rows and 3 columns.

a_{ij}

Row

Column

Example: $a_{12} = 7$

$a_{21} = 5$

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Bloom: Remembering

Types of matrices

1. Row matrix: $\begin{pmatrix} 2 & 3 & 8 \end{pmatrix}$

2. Column matrix: $\begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$

3. Zero matrix: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

4. Square matrix: $\begin{pmatrix} 1 & 4 \\ 6 & 1 \end{pmatrix}$ — 2×2 square matrix

$\begin{pmatrix} 3 & 9 & 0 \\ 4 & 6 & 8 \\ 2 & 1 & 10 \end{pmatrix}$ — 3×3 square matrix

Types of matrices

5. Diagonal matrix: $\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{pmatrix}$

6. Upper triangular matrix: $\begin{pmatrix} 3 & 5 & 1 \\ 0 & 6 & -2 \\ 0 & 0 & 8 \end{pmatrix}$

7. Lower triangular matrix: $\begin{pmatrix} 3 & 0 & 0 \\ 7 & 6 & 0 \\ 9 & 4 & 8 \end{pmatrix}$

Types of matrices

8. Identity matrix:

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$\mathbf{2} \times \mathbf{2}$ identity matrix

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$\mathbf{3} \times \mathbf{3}$ identity matrix

Operations with matrices

1. Addition

Example:

Given that $A = \begin{pmatrix} 4 & 7 & -2 \\ -3 & 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & 2 & 8 \\ -7 & -6 & -3 \end{pmatrix}$,

find $A + B$.

Solution:

$$A + B = \begin{pmatrix} 4 & 7 & -2 \\ -3 & 5 & 1 \end{pmatrix} + \begin{pmatrix} 6 & 2 & 8 \\ -7 & -6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4+6 & 7+2 & -2+8 \\ -3+(-7) & 5+(-6) & 1+(-3) \end{pmatrix}$$

Adding the corresponding elements.

$$= \begin{pmatrix} 10 & 9 & 6 \\ -10 & -1 & -2 \end{pmatrix}$$

Operations with matrices

2. Subtraction

Example:

Given that $A = \begin{pmatrix} 2 & 5 \\ 6 & 1 \\ 4 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 2 \\ 3 & -1 \\ 1 & -8 \end{pmatrix}$

find $A - B$.

Solution:

$$A - B = \begin{pmatrix} 2 & 5 \\ 6 & 1 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} -4 & 2 \\ 3 & -1 \\ 1 & -8 \end{pmatrix}$$

subtracting the corresponding elements.

$$= \begin{pmatrix} 2-(-4) & 5-2 \\ 6-3 & 1-(-1) \\ 4-1 & -1-(-8) \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 2 \\ 3 & 7 \end{pmatrix}$$

Operations with matrices

3. Scalar multiplication

Example:

If $A = \begin{pmatrix} 3 & -1 & 5 \\ -4 & 9 & 2 \end{pmatrix}$, find $2A$.

Solution:

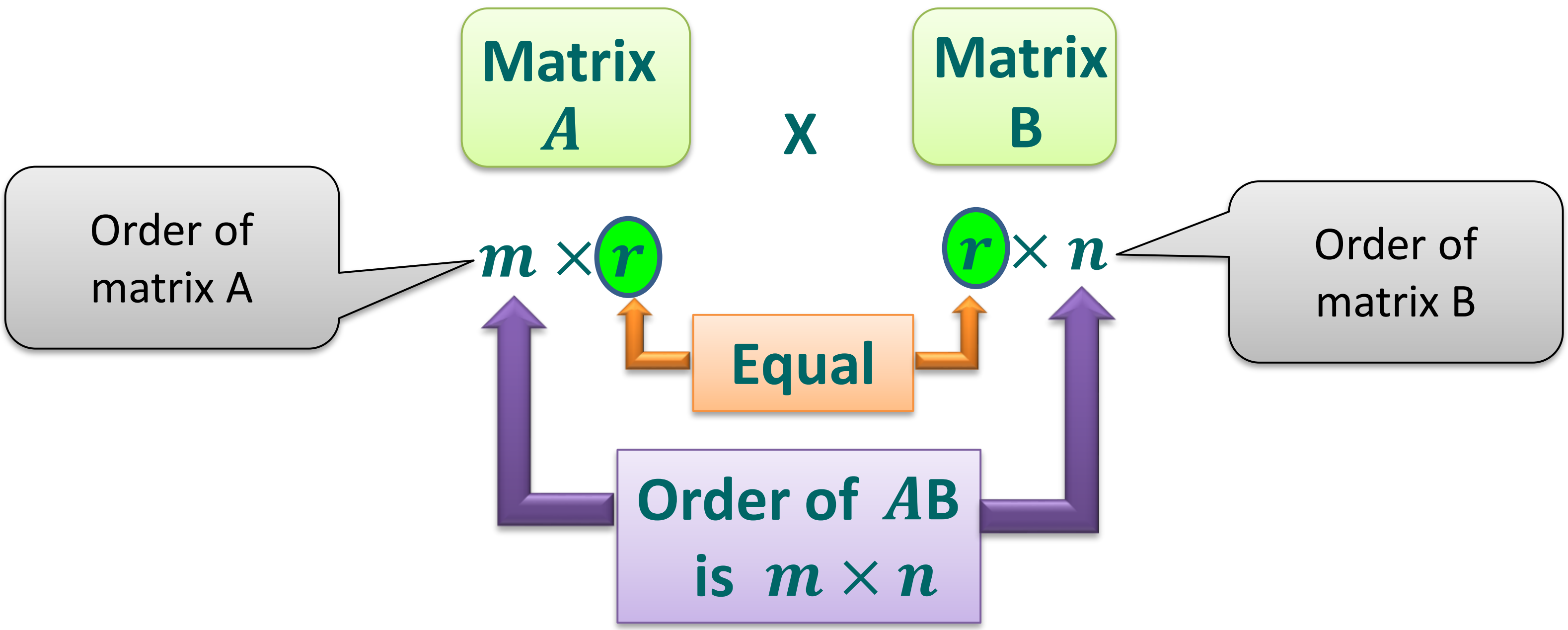
$$2A = 2\begin{pmatrix} 3 & -1 & 5 \\ -4 & 9 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(3) & 2(-1) & 2(5) \\ 2(-4) & 2(9) & 2(2) \end{pmatrix}$$ Multiplying each element of A by 2.

$$= \begin{pmatrix} 6 & -2 & 10 \\ -8 & 18 & 4 \end{pmatrix}$$

Operations with matrices

4. Multiplication of matrices



Operations with matrices

4. Multiplication of matrices

Example:

Given that $A = \begin{pmatrix} 4 & 2 & -1 \\ -3 & 0 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 4 & 5 \end{pmatrix}$,

find AB.

Solution:

$$AB = \begin{pmatrix} 4 & 2 & -1 \\ -3 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 4 & 5 \end{pmatrix}$$

Multiplying the two matrices.

$$= \begin{pmatrix} 4(1) + 2(2) + (-1)(4) & 4(3) + 2(-1) + (-1)(5) \\ -3(1) + 0(2) + (-2)(4) & -3(3) + 0(-1) + (-2)(5) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 \\ -11 & -19 \end{pmatrix}$$

Transpose of a matrix

Examples:

If $A = \begin{pmatrix} 5 & 4 & -2 \\ 1 & -6 & 8 \end{pmatrix}$, then $A^T = \begin{pmatrix} 5 & 1 \\ 4 & -6 \\ -2 & 8 \end{pmatrix}$.

If $B = \begin{pmatrix} 3 & 1 & -7 \\ 1 & 4 & 8 \\ -7 & 8 & -5 \end{pmatrix}$, then $B^T = \begin{pmatrix} 3 & 1 & -7 \\ 1 & 4 & 8 \\ -7 & 8 & -5 \end{pmatrix}$

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Bloom: Remembering

Properties of matrix transpose

1. $(A^T)^T = A$
2. $(kA)^T = kA^T$, k is a scalar
3. $(A \pm B)^T = A^T \pm B^T$
4. $(AB)^T = B^T A^T$

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Bloom: Remembering

Symmetric matrix

A symmetric matrix is a square matrix whose elements about the leading diagonal are the same.

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 3 & 4 & -3 \\ 4 & 6 & 5 \\ -3 & 5 & 10 \end{pmatrix}$$

Leading diagonal

Leading diagonal

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Bloom: Remembering

Skew-symmetric matrix

A skew-symmetric matrix is a square matrix whose elements on the leading diagonal are the zeroes whereas the elements about leading diagonal are different in signs.

Leading diagonal

$$\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix},$$

Leading diagonal

$$\begin{pmatrix} 0 & 4 & -3 \\ -4 & 0 & -5 \\ 3 & 5 & 0 \end{pmatrix}$$

Example

1. If $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 5 & 2 \\ 1 & 5 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 2 & -4 \\ 5 & 7 & 8 \\ 3 & 1 & 5 \end{pmatrix}$.

   Find (a) $\boldsymbol{A} + \boldsymbol{B}$ (b) $\boldsymbol{A} - \boldsymbol{B}$

2. If $\boldsymbol{P} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ -4 & 5 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 7 & 6 \end{pmatrix}$. Find

   (a) $\boldsymbol{2P} + \boldsymbol{Q}$ (b) $\boldsymbol{2P} - \boldsymbol{3Q}$

Example

3. If $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$.

   Calculate AB and BA.

4. If $A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 4 \\ 5 & 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 2 & 6 \\ 1 & -1 & 3 \end{pmatrix}$.

   Calculate AB.

Solution

1. (a)

$$A + B = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 5 & 2 \\ 1 & 5 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & -4 \\ 5 & 7 & 8 \\ 3 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 6 & -7 \\ 5 & 12 & 10 \\ 4 & 6 & 6 \end{pmatrix}$$

(b)

$$A - B = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 5 & 2 \\ 1 & 5 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & -4 \\ 5 & 7 & 8 \\ 3 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ -5 & -2 & -6 \\ -2 & 4 & -4 \end{pmatrix}$$

Solution (continue…)

2. (a) $2P + Q$

$$= 2\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ -4 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 6 & 0 \\ -8 & 10 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 \\ 4 & 4 \\ -1 & 16 \end{pmatrix}$$

(b) $2P - 3Q$

$$= 2\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ -4 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ -2 & 4 \\ 7 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 6 & 0 \\ -8 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ -6 & 12 \\ 21 & 18 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 4 \\ 12 & -12 \\ -29 & -8 \end{pmatrix}$$

Solution (continue…)

3. (a)

$$AB = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 3+6 & 6+(-8) \\ -1+0 & -2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -2 \\ -1 & -2 \end{pmatrix}$$

(b)

$$BA = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3+(-2) & 2+0 \\ 9+4 & 6+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 13 & 6 \end{pmatrix}$$

Solution (continue…)

4.

$$\boldsymbol{AB} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 4 \\ 5 & 7 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -5 & 2 & 6 \\ 1 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3-5+0 & 0+2-0 & 3+6+0 \\ -1-10+4 & 0+4-4 & -1+12+12 \\ 5-35+2 & 0+14-2 & 5+42+6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 2 & 9 \\ -7 & 0 & 23 \\ -28 & 12 & 53 \end{pmatrix}$$

Self-check

1. If $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ 6 & -2 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 5 & 6 & 3 \\ 4 & -1 & 0 \\ 2 & 5 & 7 \end{pmatrix}$.

   Find (a) $\boldsymbol{A} + \boldsymbol{B}$ (b) $\boldsymbol{A} - \boldsymbol{B}$

2. If $\boldsymbol{M} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 0 & -3 & 6 \end{pmatrix}$ and $\boldsymbol{N} = \begin{pmatrix} 0 & -2 & 1 \\ 5 & 6 & 3 \\ 2 & -4 & 1 \end{pmatrix}$. Find

   (a) $\boldsymbol{M} + 3\boldsymbol{N}$ (b) $3\boldsymbol{M} - 2\boldsymbol{N}$

Self-check

3. If $\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.

   Calculate $\boldsymbol{AB}$ and $\boldsymbol{BA} - 2\boldsymbol{I}$.

4. If $\boldsymbol{A} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 4 \\ 5 & 7 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ -5 & 2 & 6 \\ 1 & -1 & 3 \end{pmatrix}$.

   Calculate $\boldsymbol{BA}$.

Answer Self-check

1. (a) $A+B=\begin{pmatrix}7 & 5 & 6\\ 4 & 3 & 5\\ 8 & 3 & 8\end{pmatrix}$ **(b)** $A-B=\begin{pmatrix}-3 & -7 & 0\\ -4 & 5 & 5\\ 4 & -7 & -6\end{pmatrix}$

2. (a) $M+3N=\begin{pmatrix}1 & -4 & 6\\ 19 & 20 & 10\\ 6 & -15 & 9\end{pmatrix}$ **(b)** $3M-2N=\begin{pmatrix}3 & 10 & 7\\ 2 & -6 & -3\\ -4 & -1 & 16\end{pmatrix}$

3. $AB=\begin{pmatrix}-4 & 9\\ -3 & 4\end{pmatrix}$, $BA-2I=\begin{pmatrix}-4 & 3\\ -5 & 0\end{pmatrix}$

4. $BA=\begin{pmatrix}8 & 8 & 2\\ 13 & 41 & 20\\ 19 & 20 & 2\end{pmatrix}$

Bloom: Applying

Summary

Matrices

Types of matrices

- Row matrix
- Column matrix
- Zero matrix
- Square matrix
- Diagonal matrix
- Upper triangular matrix
- Lower triangular matrix
- Identity matrix

Operations on matrices

- Addition
- Subtraction
- Scalar multiplication
- Multiplication for 2 matrices

Transpose of a matrix

Properties of transpose

$$(A^T)^T = A$$

$$(kA)^T = kA^T$$

$$(A \pm B)^T = A^T \pm B^T$$

$$(AB)^T = B^T A^T$$

Skew-symmetric matrix

Symmetric matrix

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Bloom: Remembering

Key Terms

- Row matrix
- Column matrix
- Zero matrix
- Square matrix
- Upper triangular matrix
- Lower triangular matrix
- Identity matrix
- Transpose matrix
- Symmetric matrix
- Skew-symmetric matrix