Chapter 3: Sequences and Series

3.2 Binomial Expansion

Prepared by: kwkang

Learning Outcomes

- (a) Find the expansion of $(a + b)^n$ where n is a positive integer.
- (b) Write n! notations and ${}^{n}C_{r} = {n \choose r}$ as a binomial coefficient.
- (c) Determine the general term in a binomial expansion $(a + b)^n$ where n is a positive integer.

Learning Outcomes

- (d) Determine the expansion of $(1 + x)^n$ for |x| < 1 where *n* is a rational number for both positive and negative numbers.
 - * $(a+b)^n = a^n \left(1+\frac{a}{b}\right)^n$ where a > b
 - * Use binomial expansion to approximate values such as $\sqrt{2}$, $(1.001)^{10}$, $\sqrt[3]{5}$.

Binomial Expansion

$$n! = n(n-1)(n-2) \dots 3 \times 2 \times 1$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

If *n* is a positive integer, the general result for the expansion of $(a + b)^n$ is

Type 1

$$(a+b)^{n} = {n \choose 0} a^{n} + {n \choose 1} a^{n-1}b + {n \choose 2} a^{n-2}b^{2} + {n \choose 3} a^{n-3}b^{3} + \dots {n \choose r} a^{n-r}b^{r} + \dots + {n \choose n}b^{n}$$

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Binomial Expansion

The general term, $T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$

If *n* is negative or fractional, the binomial series for $(1 + x)^n$ is given by



$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$(a + bx)^n = a^n \left(1 + \frac{b}{a}\right)^n$$
 before applying the expansion of $(1 + x)^n$ as shown above.

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Example

- 1. Expand the following using the binomial theorem.
 - (a) $(x + y)^3$
 - (b) $(x y)^4$
- 2. Find the 12th term in the expansion of $(1 + x)^{20}$ as a series in ascending powers of x.

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Example

3. Find the coefficient of x^6 in the expansion of $(3 + x)^{12}$.

4. Find the term independent of x in the expansion of $\left(x - \frac{1}{x}\right)^{12}$.

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Solution

1. (a)
$$(x + y)^3 = x^3 + {3 \choose 1} x^2(y) + {3 \choose 2} x(y^2) + (y)^3$$

 $= x^3 + 3x^2y + 3xy^2 + y^3$
(b) $(x - y)^4 = x^4 + {4 \choose 1} x^3(-y) + {4 \choose 2} x^2(-y^2)$
 $+ {4 \choose 3} x(-y)^3 + (-y)^4$
 $= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

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2.
$$T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$$
 Using T_{r+1} term formula
 $T_{12} = {\binom{20}{11}}(1)^{20-11}(x)^{11}$
 $= 167960x^{11}$

3.
$$T_{r+1} = {\binom{12}{r}} (3)^{12-r} (x)^r \quad \text{Using } T_{r+1} \text{ term formula}$$
$$T_7 = {\binom{12}{6}} (3)^{12-6} (x)^6 \quad \text{Obviously, r=6}$$
$$= 673596 x^6$$
The coefficient of x^6 is 673596 .

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4. Independent of *x* means without *x* term. This is only make possible if the power of *x* is zero.

$$T_{r+1} = \binom{12}{r} (x)^{12-r} \left(-\frac{1}{x}\right)^{r}$$

Collect all terms in x and simplifying them if possible. $\frac{x^{12-r}}{x^r} = x^{12-2r} = x^0$ $\therefore 12 - 2r = 0$ r = 6 $So, T_7 = {\binom{12}{6}} (x)^{12-6} \left(-\frac{1}{x}\right)^6 = 924$

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Self-check

- 1. Expand the following using the binomial theorem.
 - (a) $(x + y)^4$
 - (b) $(1-x)^4$
- 2. Find the 10th term in the expansion of $(1 + 2x)^{18}$ as a series in ascending powers of x.

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Self-check

3. Find the coefficient of x^6 in the expansion of $\left(2 + \frac{x}{2}\right)^{10}$.

4. Find the term independent of x in the expansion of $\left(x + \frac{1}{2x^2}\right)^{12}$.

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Answer Self-check

(1) (a)
$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

(b) $1 - 4x + 6x^2 - 4x^3 + x^4$

$$(2) \ T_{10} = 24893440x^9$$

$$(4) \frac{495}{16}$$

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Example

- 1. Expand the following up to the term in x^3 . (a) $(1 + x)^{\frac{1}{2}}$ (b) $(1 - \frac{x}{2})^{\frac{1}{4}}$
- 2. For which value of x are the expansions in Question 1 valid?

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Example

- 3. Expand the function $(9 + x)^{\frac{1}{2}}$ in a series of ascending powers of x as far as the term in x^{3} . State the value of x for which each expansion is valid.
- 4. Obtain the expansion of $\sqrt{1-2x}$ up to the x^2 term. By putting $x = \frac{1}{100}$, find a rational approximation for $\sqrt{2}$.

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Solution

1. (a)

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^{3} + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^{2} + \frac{1}{16}x^{3} + \dots$$

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1. (b)
$$\left(1-\frac{x}{2}\right)^{\frac{1}{4}} = 1 + \frac{1}{4}\left(-\frac{x}{2}\right) + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{x}{2}\right)^{2} + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(-\frac{x}{2}\right)^{3} + \dots$$
$$= 1 - \frac{1}{8}x - \frac{3}{128}x^{2} - \frac{7}{1024}x^{3} + \dots$$

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2. (a) For expansion is valid, |x| < 1 $\therefore -1 < x < 1$

(b) For expansion is valid, $\left|\frac{x}{2}\right| < 1$ $-1 < \frac{x}{2} < 1$ $\therefore -2 < x < 2$

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3.
$$(9+x)^{\frac{1}{2}}$$

$$= \left[9\left(1+\frac{x}{9}\right)\right]^{\frac{1}{2}}$$

$$= 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$$

$$= 3\left[1+\frac{1}{2}\left(\frac{x}{9}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{9}\right)^{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{9}\right)^{3}+\dots\right]$$

$$= 3+\frac{x}{6}-\frac{1}{216}x^{2}+\frac{x^{3}}{3888}+\dots$$

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The expansion is valid for $\left|\frac{x}{9}\right| < 1$ $-1 < \frac{x}{9} < 1$ $\therefore -9 < x < 9$

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4.
$$(1-2x)^{\frac{1}{2}}$$

 $= 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-2x)^{2} + \cdots$
 $= 1 - x - \frac{1}{2}x^{2} + \cdots$
Put $x = \frac{1}{100}$ into $(1-2x)^{\frac{1}{2}}$
 $\left(1 - \frac{1}{100}\right)^{\frac{1}{2}} = 1 - \frac{1}{100} - \frac{1}{2}\left(\frac{1}{100}\right)^{2}$

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$$\sqrt{\frac{98}{100}} = 1 - \frac{1}{100} - \frac{1}{20000}$$
$$\frac{\sqrt{98}}{100} = \frac{20000 - 200 - 1}{20000}$$
$$7\sqrt{2} = \frac{19799}{2000}$$
$$\sqrt{2} = \frac{19799}{14000}$$

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Self-check

- 1. Expand the following up to the term in x^3 . (a) $(1 + x)^{\frac{1}{3}}$ (b) $(1 - \frac{x}{3})^{\frac{1}{5}}$
- 2. For which value of x are the expansions in Question 1 valid?

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Self-check

- 3. Expand the function (8 + x)^{1/3} in a series of ascending powers of x as far as the term in x³. State the value of x for which each expansion is valid.
- 4. Obtain the expansion of $\sqrt{1+x}$ up to the x^2 term. By putting $x = \frac{8}{100}$, find a rational approximation for $\sqrt{3}$.

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Answer Self-check

(1) (a)
$$1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

(b) $1 - \frac{1}{15}x - \frac{2}{225}x^2 - \frac{2}{1125}x^3$
(2) (a) $-1 < x < 1$
(b) $-3 < x < 3$

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(3)
$$2 + \frac{x}{12} - \frac{1}{288}x^2 + \frac{5}{20736}x^3 + \dots$$

-8 < *x* < **8**

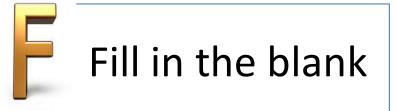
$$(4) \quad \sqrt{3} = \frac{433}{250}$$

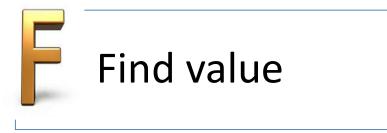
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BFF's Technique







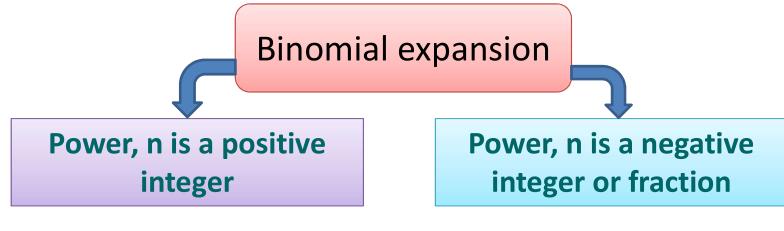
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Bloom: Remembering

BFF's Technique

	n positive integer	n negative integer or fraction
Formula	$(x+a)^n = \sum_{k=0}^n \binom{n}{k} x^k a^{n-k}$	$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$
BFF's technique	 (2x + 5)² 3 terms if power to 2 4 terms if power to 3 and so on 	$(1+x)^{-2}$ $(a+bx)^{n} = a^{n} \left(1 + \frac{b}{a}\right)^{n}$ Must in the form of $(1+x)^{n}$ before doing expansion.
Step 1: Bracket	()()()+()()+()()()+()())	$1 + \frac{()}{1!} ()^{1} + \frac{()()}{2!} ()^{2} + \frac{()()()}{3!} ()^{3} + \cdots$
Step 2: Fill in the blank	$\binom{2}{0}(2x)^2(5)^0 + \binom{2}{1}(2x)^1(5)^1 + \binom{2}{2}(2x)^0(5)^2$	$1 + \frac{(-2)}{1!}(x)^{1} + \frac{(-2)(-2-1)}{2!}(x)^{2} + \frac{(-2)(-2-1)(-2-2)}{3!}(x)^{3} + \cdots$
Step 3: Find value	$4x^2 + 20x + 25$	$1-2x+3x^2-4x^3+\cdots$

Summary



$$(a+b)^{n} = {\binom{n}{0}}a^{n} + {\binom{n}{1}}a^{n-1}b + {\binom{n}{2}}a^{n-2}b^{2} + {\binom{n}{3}}a^{n-3}b^{3} + \dots$$

$$T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$$

 $(1+x)^{n} = 1 + nx + \frac{n(n-1)x^{2}}{2!} + \frac{n(n-1)(n-2)x^{3}}{3!} + \dots$ $(a+bx)^{n} = a^{n} \left(1 + \frac{b}{a}\right)^{n}$ before applying the expansion of $(1+x)^{n}$ as shown above.

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Bloom: Remembering

Key Terms

- Binomial expansion
- Validity of expansion
- Term
- Coefficient
- General term
- Positive integer
- Fraction
- Rational number

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