Chapter 3: Sequences and Series

3.2 Binomial Expansion

Prepared by: kwkang

Learning Outcomes

(a) Find the expansion of $(a+b)^n$ where n is a positive integer.

(b) Write $n!$ notations and ${}^nC_r = \binom{n}{r}$ as a binomial coefficient.

(c) Determine the general term in a binomial expansion $(a+b)^n$ where n is a positive integer.

Bloom: Understanding

Learning Outcomes

(d) Determine the expansion of $(1+x)^n$ for $|x|<1$ where n is a rational number for both positive and negative numbers.

* $(a+b)^n = a^n\left(1+\frac{a}{b}\right)^n$ where $a>b$
* Use binomial expansion to approximate values such as $\sqrt{2}, (1.001)^{10}, \sqrt[3]{5}$.

Binomial Expansion

$$n! = n(n-1)(n-2)\dots 3\times 2\times 1$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

If n is a positive integer, the general result for the expansion of $(a+b)^n$ is

Type 1

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n$$

Binomial Expansion

The general term, $T_{r+1} = {}^{n}C_{r} a^{n-r} b^{r}$

If n is **negative or fractional**, the binomial series for $(1+x)^n$ is given by

Type 2

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$(a+bx)^n = a^n\left(1+\frac{b}{a}\right)^n$$

before applying the expansion of $(1+x)^n$ as shown above.

Example

1. Expand the following using the binomial theorem.

   (a) $(x+y)^3$

   (b) $(x-y)^4$

2. Find the 12$^{\text{th}}$ term in the expansion of $(1+x)^{20}$ as a series in ascending powers of x .

Example

3. Find the coefficient of x^6 in the expansion of $(3+x)^{12}$.

4. Find the term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{12}$.

Bloom: Understanding

Solution

1. (a) $(x+y)^3 = x^3 + \binom{3}{1}x^2(y) + \binom{3}{2}x(y^2) + (y)^3$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

(b) $(x-y)^4 = x^4 + \binom{4}{1}x^3(-y) + \binom{4}{2}x^2(-y^2)$

$$+ \binom{4}{3}x(-y)^3 + (-y)^4$$

$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

Solution (continue…)

2. $T_{r+1} = {}^{n}C_r a^{n-r} b^r$ Using T_{r+1} term formula

$$T_{12} = \binom{20}{11}(1)^{20-11}(x)^{11}$$

$$= 167960x^{11}$$

3. $T_{r+1} = \binom{12}{r}(3)^{12-r}(x)^r$ Using T_{r+1} term formula

$T_7 = \binom{12}{6}(3)^{12-6}(x)^6$ Obviously, r=6

$$= 673596x^6$$

The coeficient of x^6 is 673596 .

Solution (continue…)

4. **Independent of x means without x term. This is only make possible if the power of x is zero.**

$$T_{r+1} = \binom{12}{r}(x)^{12-r}\left(-\frac{1}{x}\right)^{r}$$

Collect all terms in x and simplifying them if possible.

$$\frac{x^{12-r}}{x^{r}} = x^{12-2r} = x^{0}$$

$$\therefore 12 - 2r = 0$$

$$r = 6$$

$$So,\ \ T_7 = \binom{12}{6}(x)^{12-6}\left(-\frac{1}{x}\right)^{6} = 924$$

Self-check

1. Expand the following using the binomial theorem.

   (a) $(x+y)^4$

   (b) $(1-x)^4$

2. Find the 10th term in the expansion of $(1+2x)^{18}$ as a series in ascending powers of x .

Self-check

3. Find the coefficient of x^6 in the expansion of $\left(2+\frac{x}{2}\right)^{10}$.

4. Find the term independent of x in the expansion of $\left(x+\frac{1}{2x^2}\right)^{12}$.

Bloom: Applying

Answer Self-check

(1) (a) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

(b) $1 - 4x + 6x^2 - 4x^3 + x^4$

(2) $T_{10} = 24893440x^9$

(3) 52.5

(4) $\frac{495}{16}$

Bloom: Applying

Example

1. Expand the following up to the term in x^3.

   (a) $(1+x)^{\frac{1}{2}}$

   (b) $\left(1-\frac{x}{2}\right)^{\frac{1}{4}}$

2. For which value of x are the expansions in Question 1 valid?

Bloom: Understanding

Example

3. Expand the function $(9 + x)^{\frac{1}{2}}$ in a series of ascending powers of x as far as the term in x^3. State the value of x for which each expansion is valid.

4. Obtain the expansion of $\sqrt{1 - 2x}$ up to the x^2 term. By putting $x = \frac{1}{100}$, find a rational approximation for $\sqrt{2}$.

Bloom: Understanding

Solution

1. (a)

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ \dots$$

Solution (Continue...)

1. (b)

$$\left(1-\frac{x}{2}\right)^{\frac{1}{4}} = 1+\frac{1}{4}\left(-\frac{x}{2}\right)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(-\frac{x}{2}\right)^2 + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(-\frac{x}{2}\right)^3 + \ldots$$

$$= 1-\frac{1}{8}x-\frac{3}{128}x^2-\frac{7}{1024}x^3+\ \ldots$$

Solution (Continue…)

2. (a) For expansion is valid,

$$|x| < 1$$

$$\therefore -1 < x < 1$$

(b) For expansion is valid,

$$\left|\frac{x}{2}\right| < 1$$

$$-1 < \frac{x}{2} < 1$$

$$\therefore -2 < x < 2$$

Solution (Continue…)

3. $(9+x)^{\frac{1}{2}}$

$$= \left[9\left(1+\frac{x}{9}\right)\right]^{\frac{1}{2}}$$

$$= 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}}$$

$$= 3\left[1+\frac{1}{2}\left(\frac{x}{9}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{9}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{9}\right)^3+\ \ldots\right]$$

$$= 3+\frac{x}{6}-\frac{1}{216}x^2+\frac{x^3}{3888}+\cdots$$

Bloom: Understanding

Solution (Continue…)

The expansion is valid for

$$\left|\frac{x}{9}\right| < 1$$

$$-1 < \frac{x}{9} < 1$$

$$\therefore -9 < x < 9$$

Solution (Continue…)

4. $(1-2x)^{\frac{1}{2}}$

$$= 1 + \frac{1}{2}(-2x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-2x)^2 + \cdots$$

$$= 1 - x - \frac{1}{2}x^2 + \ \ldots$$

Put $x = \frac{1}{100}$ **into** $(1-2x)^{\frac{1}{2}}$

$$\left(1-\frac{1}{100}\right)^{\frac{1}{2}} = 1 - \frac{1}{100} - \frac{1}{2}\left(\frac{1}{100}\right)^2$$

Solution (Continue…)

$$\sqrt{\frac{98}{100}} = 1 - \frac{1}{100} - \frac{1}{20000}$$

$$\frac{\sqrt{98}}{100} = \frac{20000 - 200 - 1}{20000}$$

$$7\sqrt{2} = \frac{19799}{2000}$$

$$\sqrt{2} = \frac{19799}{14000}$$

Self-check

1. Expand the following up to the term in x^3.

   (a) $(1+x)^{\frac{1}{3}}$

   (b) $\left(1-\frac{x}{3}\right)^{\frac{1}{5}}$

2. For which value of x are the expansions in Question 1 valid?

Self-check

3. Expand the function $(8+x)^{\frac{1}{3}}$ in a series of ascending powers of x as far as the term in x^3. State the value of x for which each expansion is valid.

4. Obtain the expansion of $\sqrt{1+x}$ up to the x^2 term. By putting $x=\frac{8}{100}$, find a rational approximation for $\sqrt{3}$.

Bloom: Applying

Answer Self-check

(1) (a) $\mathbf{1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3}$

(b) $\mathbf{1 - \frac{1}{15}x - \frac{2}{225}x^2 - \frac{2}{1125}x^3}$

(2) (a) $\mathbf{-1 < x < 1}$

(b) $\mathbf{-3 < x < 3}$

kwkang@KMK

Bloom: Applying

Answer Self-check

(3) $2+\frac{x}{12}-\frac{1}{288}x^2+\frac{5}{20736}x^3+\ \dots$

$-8<x<8$

(4) $\sqrt{3}=\frac{433}{250}$

BFF's Technique

- **B** Bracket
- **F** Fill in the blank
- **F** Find value

BFF's Technique

	n positive integer	n negative integer or fraction
Formula	$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^k a^{n-k}$	$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$
BFF's technique	$(2x+5)^2$ • 3 terms if power to 2 • 4 terms if power to 3 and so on…	$(1+x)^{-2}$ $(a+bx)^n = a^n\left(1+\frac{b}{a}\right)^n$ Must in the form of $(1+x)^n$ before doing expansion.
Step 1: Bracket	$(\ \ )(\ \ )(\ \ ) + (\ \ )(\ \ )(\ \ ) + (\ \ )(\ \ )(\ \ )$	$1 + \frac{(\ \ )}{1!}(\ \ )^1 + \frac{(\ \ )(\ \ )}{2!}(\ \ )^2 + \frac{(\ \ )(\ \ )(\ \ )}{3!}(\ \ )^3 + \cdots$
Step 2: Fill in the blank	$\binom{2}{0}(2x)^2(5)^0 + \binom{2}{1}(2x)^1(5)^1 + \binom{2}{2}(2x)^0(5)^2$	$1 + \frac{(-2)}{1!}(x)^1 + \frac{(-2)(-2-1)}{2!}(x)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}(x)^3 + \cdots$
Step 3: Find value	$4x^2 + 20x + 25$	$1 - 2x + 3x^2 - 4x^3 + \cdots$

Summary

Binomial expansion

Power, n is a positive integer

$$(a+b)^{n} = \binom{n}{0}a^{n} + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^{2} + \binom{n}{3}a^{n-3}b^{3} + \ldots$$

$$T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$$

Power, n is a negative integer or fraction

$$(1+x)^{n} = 1 + nx + \frac{n(n-1)x^{2}}{2!} + \frac{n(n-1)(n-2)x^{3}}{3!} + \ \ldots$$

$$(a+bx)^{n} = a^{n}\left(1+\frac{b}{a}\right)^{n}$$

before applying the expansion of $(1+x)^{n}$ as shown above.

kwkang@KMK

Bloom: Remembering

Key Terms

- Binomial expansion
- Validity of expansion
- Term
- Coefficient
- General term
- Positive integer
- Fraction
- Rational number