Chapter 3: Sequences and Series

3.1 Sequences and series

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Learning Outcomes

(a) Write nth term of simple sequences and series.

(b) Find the nth term of arithmetic sequence and series, $T_n = a + (n-1)d$ use the sum formula, $S_n = \frac{n}{2}[2a + (n-1)d]$ and $S_n = \frac{n}{2}(a + l)$.

(c) Find the nth term of geometric sequences and series, $T_n = ar^{n-1}$ and use the sum formula, $S_n = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$.

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Bloom: Understanding

Sequences and series

Sequences

- A sequence is a set of numbers occurring in a definite order. The numbers are produced according to a particular rule.
  Example: (i) $1, 3, 5, 7$ (finite sequence)
  (ii) $1, 3, 5, 7, \ldots$ (infinite sequence)
- Each member of a sequence is called a term.

Series

- A series is the sum of the terms of a sequence.
  Example: (i) $1 + 3 + 5 + 7$ (finite series)
  (ii) $1 + 3 + 5 + 7 + \cdots$ (infinite series)

Example

n th term of simple sequences

Example 1: Write the general term for the finite sequence.

$$2, 4, 8, 16$$

Solution:

$$a_1 = 2^1 = 2$$
$$a_2 = 2^2 = 4$$
$$a_3 = 2^3 = 8$$
$$a_4 = 2^4 = 16$$

$$\therefore a_n = 2^n$$

Arithmetic Series

$a, a+d, a+2d, \ldots, a+(n-1)d$

where a is the **first term** and d is **common difference**

n th term of an arithmetic sequence

$$T_n = a + (n-1)d$$

Sum to n term of an arithmetic sequence

$$S_n = \frac{n}{2}[2a + (n-1)d] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

where l is the last term.

Example

1. Find the $\boldsymbol{n}$th term of the arithmetic sequence

   $\mathbf{8, 12, 16, \ldots}$

2. The 3rd term of an arithmetic sequence is 16 and the 13th term is 46. Find the first term and the common difference.

3. The 7th term of an arithmetic sequence is five times the second term. The two terms differ by 20. Find the first term and the common difference.

Example

4. Find the sum of the arithmetic series.

$$\mathbf{3 + 5 + 7 + 9 + \ldots + 41}$$

5. Given that the 1^{st} and 18^{th} terms of an arithmetic progression are 2 and 53 respectively, find the 90^{th} term and the sum of the first 50 terms.

6. How many terms of the arithmetic series $\mathbf{1 + 2 + 3 + 4 +}$ …required to make a sum of 210?

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Solution

1. $a = 8, d = -4$

$T_n = a + (n-1)d$ Using the *n*th term formula.

$T_n = (8) + (n-1)(-4)$

$= 12 - 4n$

2. $T_3 = 16, \quad T_{13} = 46,$

$T_n = a + (n-1)d$ Using the *n*th term formula.

$a + (2d) = 16$ .... (1)

$a + (12d) = 46$ .... (2)

$(2) - (1): \quad 10d = 30$

$\therefore d = 3$

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Solution (continue…)

From (1): $when\ d = 3$,

$$a + (2(3)) = 16$$

$$a = 10$$

Therefore the first term is 10 and the common difference is 3.

3. $T_7 = 5T_2$

$$a + 6d = 5[a + d]$$

$$d = 4a \quad \text{.......... (1)}$$

$$T_7 - T_2 = 20$$

$$a + 6d - (a + d) = 20$$

$$5d = 20$$

$$\therefore d = 4$$

Substitute $d = 4$ into (1):

$$(4) = 4a$$

$$\therefore a = 1$$

Therefore the first term is 1 and the common difference is 4.

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Solution (continue...)

4. Let $T_n = 41, a = 3, d = 2$

$$T_n = a + (n-1)d$$

Using the nth term formula.

$$3 + (n-1)2 = 41$$

$$n = 20$$

$$S_n = \frac{n}{2}(a + l)$$

Using the sum formula.

$$S_{20} = \frac{20}{2}(3 + 41)$$

$$= 440$$

Solution (continue…)

5. Given $a = 2, T_{18} = 53$

$$2 + (18 - 1)d = 53$$

$$d = 3$$

Using the *n*th term formula.

Then $T_{90} = 2 + (90 - 1)(3)$

$$= 269$$

$$S_{50} = \frac{50}{2}[2(2) + (50 - 1)(3)]$$

$$= 3775$$

Using the sum formula.

Solution (continue...)

6. Given $S_n = 210$

$$\frac{n}{2}[2 + (n-1)(1)] = 210$$ Using the sum formula.

$$n(n+1) = 420$$

$$n^2 + n - 420 = 0$$ Expanding and rearranging RHS=0.

$$(n+21)(n-20) = 0$$ Factorizing quadratic equation.

$$n = -21, n = 20$$

n represents the number of terms, so it should be a natural number. Therefore $n = 20$.

Geometric Series

$a, ar, ar^2, ar^3, \dots, ar^{n-1}, \dots$

where a is the first term and d is common ratio.

n th term of an geometric sequence

$$T_n = ar^{n-1}$$

Sum to n term of a geometric progression

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{where } r \neq 1$$

Example

1. Write down the nth term of the geometric sequence $\mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{16}, \ldots$.

2. Find the first term and common ratio of a geometric sequence, given that the sixth term is 486 and the third term is 18.

3. Find the numbers of terms in the geometric sequence $\mathbf{1, 2, 4, 8, \ldots\ldots, 131072}$

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Example

4. Find the sum of the first nine terms of the geometric sequence $\mathbf{2, 6, 18, 54, \ldots}$

5. Find the sum of the first twelve terms of a geometric sequence that has a first term of $\mathbf{\frac{1}{9}}$ and an 8th term of 243.

3. How many terms of sequence $\mathbf{1, 2, 4, 8, \ldots\ldots}$ are required to give a sum of 16383?

Solution

1. $a = 2, r = 2$

$T_n = ar^{n-1}$

$T_n = (2)(2)^{n-1}$

$= 2^n$

Using the nth term formula.

2. $T_6 = ar^5 = 486$ .... (1)

$T_3 = ar^2 = 18$ .... (2)

Using the nth term formula.

$\frac{(1)}{(2)}: \quad \frac{ar^5}{ar^2} = \frac{486}{18}$

$r^3 = 27$

$r = 3$

From (2): when $r = 3$,

$a(3)^2 = 18$

$a = 2$

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Solution (Continue…)

3. $a = 1, r = 2, T_n = 131072$

$T_n = ar^{n-1}$ Using the *n*th term formula.

$131072 = (1)(2)^{n-1}$

$131072 = 2^{n-1}$

$log_{10}131072 = log_{10}2^{n-1}$ Taking logarithms of both sides of the equation.

$$n - 1 = \frac{log_{10}131072}{log_{10}2}$$

$n - 1 = 17$

$n = 18$

So, the sequence has 18 terms.

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Solution (Continue…)

4. $a = 2, r = 3, n = 9$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Using the sum to *n*th term formula.

$$S_9 = \frac{2\left(1 - (3)^9\right)}{1 - (3)}$$

$$S_9 = 19682$$

Solution (Continue…)

5. $$a = \frac{1}{9}, T_8 = 243$$

Using the *n*th term formula,

$$T_8 = 243$$

$$ar^7 = 243$$

$$\frac{1}{9}r^7 = 243$$

$$r^7 = 2187$$

$$r = 3$$

Using the sum to *n*th term formula,

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{12} = \frac{\frac{1}{9}\left(1 - (3)^{12}\right)}{1 - (3)}$$

$$S_{12} = \frac{265720}{9}$$

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Solution (Continue…)

6. $a = 1, r = 2, S_n = 16383$

Using the sum to *n*th term formula,

$$16383 = \frac{a(1-r^n)}{1-r}$$

$$16383 = \frac{1(1-2^n)}{1-2}$$

$$-16383 = (1-2^n)$$

$$2^n = 16384$$

Taking logarithms of both sides of the equation,

$$log2^n = log16384$$

$$nlog2 = log16384$$

$$n = \frac{log16384}{log2}$$

$$n = 14$$

Self-check

(1) Find the nth term of the following sequences.

(a) $3, 6, 9, 12, \ldots$ (b) $3, 9, 27, 81, \ldots$

(2) Write down the nth term of $-6, -4, -2, \ldots$.

(3) The 4th term of an arithmetic sequence is -5 and the 15th term is -49. Find the first term and the common difference.

Self-check

(4) The 9th term of an arithmetic sequence is six times the fourth term. The two terms differ by 25. Find the first term and the common difference.

(5) Find the sum of $\mathbf{80} + \mathbf{75} + \mathbf{70} + \ \dots \ - \mathbf{10}$.

(6) Given that the 2nd and 20th terms of an arithmetic progression are -5 and -41 respectively, find the 80th term and the sum of the first 40 terms.

Bloom: Applying

Self-check

(7) How many terms of the arithmetic series $\mathbf{2+4+6+8}\ldots$ required to make a sum of 3080?

(8) Write down the $\boldsymbol{n}$th term of $\mathbf{\frac{1}{4}, \frac{1}{2} 1, 2, \ldots}$.

(9) In a geometric sequence with positive terms, the third term is $\mathbf{\frac{1}{4}}$ and the seventh term is $\mathbf{\frac{1}{64}}$. Find the first term and the common ratio.

Bloom: Applying

Self-check

(10) Find the number of terms in the geometric sequence $\mathbf{1, 1.1, 1.21, \ldots, 1.771561}$.

(11) Find the sum of the first eleven terms of

$\mathbf{3, 6, 12, 24, \ldots}$.

(12) Find the sum of the first nine terms of a geometric sequence that has a fourth term of $-\frac{\mathbf{1}}{\mathbf{8}}$ and a seventh term of $\frac{\mathbf{1}}{\mathbf{512}}$.

Bloom: Applying

Self-check

(13) How many terms of the sequence $\mathbf{125, 25, 5, 1, \ldots}$ are required to give a sum of $\mathbf{\frac{97656}{625}}$.

Bloom: Applying

Answer Self-check

(1) (a) $\boldsymbol{3n}$ (b) $\boldsymbol{3^n}$

(2) $\boldsymbol{T_n = -8 + 2n}$

(3) $\boldsymbol{a = 7, d = -4}$

(4) $\boldsymbol{a = -10, d = 5}$

(5) 665

(6) ,$\boldsymbol{T_{80} = -161}$ $\boldsymbol{S_{40} = -1680}$

(7) 55

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Bloom: Applying

Answer Self-check

(8) $T_n = 2^{n-3}$

(9) $a = 1, r = \frac{1}{2}$

(10) 7

(11) 6141

(12) $\frac{52429}{8192}$

(13) $n = 8$

Bloom: Applying

HOTS

<u>Question:</u>

The sum of three numbers in a particular arithmetic sequence is 3 and their product is -15. Then, find the numbers that satisfied the sequence.

HOTS

Solution:

Let the sequence represented by a, b, c

So, we write $b - a = c - b$

$2b = a + c$ ....(1) Rearranging equation

The sum of 3 numbers,

$a + b + c = 3$ ....(2)

The product of 3 numbers,

$abc = -15$ ....(3)

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Bloom: Analyzing

HOTS

From (2): $(a + c) + b = 3$ ....(4) **Rearranging equation**

Substitute (1) into (4):

$$(2b) + b = 3$$
$$b = 1$$

Substitute $b = 1$ into (2) & (4):

$$a + c = 2$$
$$c = 2 - a \quad \text{....(5)}$$
$$ac = -15 \quad \text{....(6)}$$

Rearranging equation

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Bloom: Analyzing

HOTS

Substitute (5) into (6):

$$a(2-a)=-15$$

$$a^2-2a-15=0$$

$$(a-5)(a+3)=0$$

$a=5$ or $a=-3$

From (5): $c=2-(5)=-3$

$$c=2-(-3)=5$$

Therefore, the sequence could be either

$5, 1, -3$ or $-3, 1, 5$.

Summary

Sequences and Series

Arithmetic series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_n = \frac{n}{2}(a + l)$$

Geometric series

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

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Bloom: Remembering

Key Terms

- Sequences
- Series
- Arithmetic series
- Geometric series
- Common difference
- Common ratio

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