Chapter 2: Equations, Inequalities and Absolute Values

2.1 Equations

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Learning Outcomes

(a) Find the equations involving surds, indices and logarithms

** Single equations such as* $2^x = 8$ *,*

$4^x + 2 = 3(2)^x$ *and equations that result to quadratic forms (solve by using factorisation, completing the square and formula).*

*** Exclude solving the simultaneous equations.*

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Equation of Indices

Solving equation of indices also known as solving exponential equations

Methods of solving:

Method 1: Change to same base.

Eg: $4^{5x+2} = 8$

Method 2: Substitution.

Eg: $3^{2x+1} - 5\left(3^{x}\right) + 2 = 0$

Bloom: Remembering

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Example

Solve.

(a) $2^{5x+2} = 8$

(b) $7^{3x-8} = 3^{x}$

(c) $2e^{2x} - 7e^{x} - 15 = 0$

Bloom: Understanding

Solution

(a) $2^{5x+2} = 8$

$2^{5x+2} = 2^3$ **Rewrite the problem using the same base.**

$5x + 2 = 3$ **Since the base are the same, we can drop the bases and set the exponents equal to each other.**

$x = \dfrac{1}{5}$ **Finish solving the problem by subtracting 2 from each side and then dividing each side by 5**

Hint

Solving Exponential Equations with the Same Base

Step 1: Determine if the numbers can be written using the same base.
If so, go to Step 2.
If not, stop and use Steps for Solving an Exponential Equation with Different Bases.

Step 2: Rewrite the problem using the same base.

Step 3: Use the properties of exponents to simplify the problem.

Step 4: Once the bases are the same, drop the bases and set the exponents equal to each other.

Step 5: Finish solving the problem by isolating the variable.

Bloom: Remembering

Solution

(b) $7^{3x-8} = 3^{x}$

$$\log(7^{3x-8}) = \log(3^{x})$$

Take the common logarithm or natural logarithm of each side.

$$(3x-8)\log 7 = x\log 3$$

Use the properties of logarithms to rewrite the problem. Move the exponent out front which turns this into a multiplication problem.

Solution (continue...)

$$3x\log 7 - 8\log 7 = x\log 3$$

Expanding the left hand side of equation.

$$3x\log 7 - x\log 3 = 8\log 7$$

Collecting the x terms.

$$x(3\log 7 - \log 3) = 8\log 7$$

Factoring out the x.

$$x = \frac{8\log 7}{3\log 7 - \log 3}$$

Evaluating from a calculator.

$$x = 3.285$$

Hint

Solving Exponential Equations with Different Base

Step 1: Determine if the numbers can be written using the same base.
If so, stop and use Steps for Solving an Exponential Equation with the Same Base.
If not, go to Step 2.

Step 2: Take the common logarithm or natural logarithm of each side.

Step 3: Use the properties of logarithms $\log_a m^p = p \log_a m$ to rewrite the problem.

Step 4: Divide each side by the logarithm.

Step 5: Use a calculator to find the decimal approximation of the logarithms.

Step 6: Finish solving the problem by isolating the variable.

Solution

(b) $2e^{2x} - 7e^{x} - 15 = 0$

$2(e^{x})^{2} - 7e^{x} - 15 = 0$

Use the rules of indices to rewrite the problem $e^{2x} = (e^{x})^{2}$.

$2y^{2} - 7y - 15 = 0$

Substituting $y = e^{x}$.

$(2y+3)(y-5) = 0$

Factorising $2y^{2} - 7y - 15$.

$2y + 3 = 0$ **or** $y - 5 = 0$

$y = -\frac{3}{2}$ **or** $y = 5$

Solve the eqautions by isolating the variables.

Solution (Continue...)

When $y = -\dfrac{3}{2}$ **,** $e^x = -\dfrac{3}{2}$

Since $e^x > 0$ **for all values of** x **,** $e^x = -\dfrac{3}{2}$ **has no solutions.**

When $y = 5$ **,** $e^x = 5$

$$x = \ln 5$$

Finish solving by taking the natural logarithm of each side.

Learning Tip: $\ln e^x = x$

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Bloom: Understanding

Equation of Surd

Steps of Solving Equation of Surd

Step 1: Check whether need to transpose term.

If no, proceed to Step 2.

Eg: $\sqrt{x+2} = 5x+3$

If yes, transpose.

Eg: $\sqrt{x+4} + \sqrt{x} = 2$ tranpose $\sqrt{x+4} = 2 - \sqrt{x}$

Step 2: Squaring both sides of the equation.

Step 3: Finish solving the problem by isolating the variable.

Step 4: Check your roots by substituting answers into the original equation.

Example

Solve the following equations.

(a) $\sqrt{6-x}=x+6$

(b) $\sqrt{4x+9}-\sqrt{2x+1}=2$

Bloom: Understanding

Solution

(a) $\sqrt{6-x} = x+6$

$\left(\sqrt{6-x}\right)^2 = (x+6)^2$ **Squaring the both sides of the equation.**

$6 - x = x^2 + 12x + 36$ **Expanding.**

$x^2 + 13x + 30 = 0$ **Writing quadratic equation in the general form.**

$(x+3)(x+10) = 0$ **Factorising.**

$x = -3$ **or** $x = -10$

Solution (Continue…)

Check: $\sqrt{6-(-3)} = (-3)+6$

$$\sqrt{9} = 3$$

$$3 = 3$$

$\therefore x = -3$ **is a solution.**

$$\sqrt{6-(-10)} = (-10)+6$$

$$\sqrt{16} = -4$$

$$4 \neq -4$$

$\therefore x = -10$ **is rejected.**

Check your roots by substituting answers into the original equation.

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Bloom: Understanding

Solution

(b) $\sqrt{4x+9}-\sqrt{2x+1}=2$

$\sqrt{4x+9}=2+\sqrt{2x+1}$ **Tranpose.**

$\left(\sqrt{4x+9}\right)^2=\left(2+\sqrt{2x+1}\right)^2$ **Squaring the both sides of the equation.**

$4x+9=4+4\sqrt{2x+1}+2x+1$ **Expanding.**

$2x+4=4\sqrt{2x+1}$ **Rearrange the equation.**

$(2x+4)^2=\left(4\sqrt{2x+1}\right)^2$ **Squaring the both sides of the equation.**

Solution (Continue...)

$$4x^2 + 16x + 16 = 16(2x+1)$$

Expanding.

$$4x^2 - 16x = 0$$

Writing quadratic equation in the general form.

$$4x(x-4) = 0$$

Factorising.

$x = 0$ **or** $x = 4$

Solution (Continue…)

Check: $\sqrt{4(0)+9}-\sqrt{2(0)+1}=2$

$$\sqrt{9}-\sqrt{1}=2$$

$$2=2$$

$\therefore x=0$ **is a solution.**

$$\sqrt{4(4)+9}-\sqrt{2(4)+1}=2$$

$$\sqrt{25}-\sqrt{9}=2$$

$$2=2$$

$\therefore x=4$ **is a solution.**

Check your roots by substituting answers into the original equation.

Equation of Logarithms

Methods of Solving

If there are logs on ONE side

Step 1: Isolate the logarithm with same base.

Step 2: Use log properties to condense to one log.

Step 3: Convert to exponential/index form.

Step 4: Solve and check answers by substituting answers into the original equation.

If there are logs on BOTH side

Step 1: Use log properties to condense each side to one log with same base.

Step 2: Set the log arguments equal to each other.

Step 3: Solve and check answers by substituting answers into the original equation.

Example

Solve

(a) $\log(x+2)+\log(x-1)=1$

(b) $\ln(4x+1)-\ln(3x-2)=\ln 5$

Bloom: Understanding

Solution

(a) $\log(x+2)+\log(x-1)=1$ **Log on one side.**

$\log(x+2)(x-1)=1$ **Use log property to condense to one log.**

$(x+2)(x-1)=10^1$ **Convert to index form.**

$x^2+x-12=0$ **Writing quadratic equation in the general form.**

$(x-3)(x+4)=0$ **Factorising.**

$x=3$ **or** $x=-4$

Solution (Continue...)

Check: $\log((3)+2)+\log((3)-1)$

$= \log 5 + \log 2$

$= \log 10 = 1$

$\therefore x = 3$ **is a solution.**

$\log((-4)+2)+\log((-4)-1)$

$= \log(-2)+\log(-5)$

Since $\log(-2)$ **and** $\log(-5)$ **are not defined,** $x=-4$ **does not satisfy the equation.**

Therefore $x=3$ **is the equation.**

Check your answer by substituting answers into the original equation.

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Bloom: Understanding

Solution

(b) $\ln(4x+1)-\ln(3x-2)=\ln 5$ **Log on both side.**

$$\ln\left(\frac{4x+1}{3x-2}\right)=\ln 5$$

Use log property to condense to each side to one log.

$$\frac{4x+1}{3x-2}=5$$

If $\ln p=\ln q$ **, then** $p=q$ **.**

$$4x+1=5(3x-2)$$

Solving.

$$4x+1=15x-10$$

$$x=1$$

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Bloom: Understanding

Solution (Continue…)

Check: $\ln(4(1)+1)-\ln(3(1)-2)$

$= \ln(5)-\ln(1)$

$= \ln 5$

$\therefore x = 1$ **is a solution.**

Check your answer by substituting answer into the original equation.

Self-check

(1) Solve the following equations without using a calculator.

(a) $2^{5x+2}=8$ (b) $5^{x+8}=15^{2x+3}$

(2) Solve the equation $9e^{2x}-22e^{x}-15=0$.

(3) Solve the following equations.

(a) $\sqrt{x-2}+x=8$ (b) $\sqrt{x}+\sqrt{x+4}=2$

Self-check

(4) Solve the following equations.

(a) $\log_2(x+5)+2=\log_2(10x+2)$

(b) $\log_3 x+12\log_x 3=8$

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Bloom: Applying

Answer Self-check

(1) (a) $\frac{1}{5}$ (b) 1.248

(2) $x = \ln 3$

(3) (a) 6 (b) 0

(4) (a) 3 (b) $x = 9$ **or** $x = 729$

Bloom: Applying

Summary

Surd

Method of solving:

- Squaring both sides

Equation

Logarithms

Method of solving:

1. Logs on one side
- Isolate log with same base
- Condense to one log (use properties)
- Convert to index form and solve

2. Logs on both sides
- Condense each side to one log (use properties)
- Eliminate log on both sides
- Solve

Indices

Method of solving:

1. Change to same base
2. Substitution

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Bloom: Remembering

Key Terms

- Indeces
- Surds
- Logarithms
- Solve equation
- Change base
- Squaring both sides
- Same base equal the power
- Check answer

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