Chapter 1: Number System

1.3 Indices, Surds and Logarithms

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Learning Outcomes

(a) Express the rules of indices

(b) Explain the meaning of a surd and its conjugate

(c) Perform algebraic operations on surds

(d) Express the laws of logarithms such as:

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(iii) $\log_a M^N = N \log_a M$

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Learning Outcomes

(d) Express the laws of logarithms such as:

(i) $\log_a MN = \log_a M + \log_a N$

(ii) $\log_a \dfrac{M}{N} = \log_a M - \log_a N$

(iii) $\log_a M^N = N \log_a M$

(e) Change the base of logarithm using

$$\log_a M = \frac{\log_b M}{\log_b a}$$

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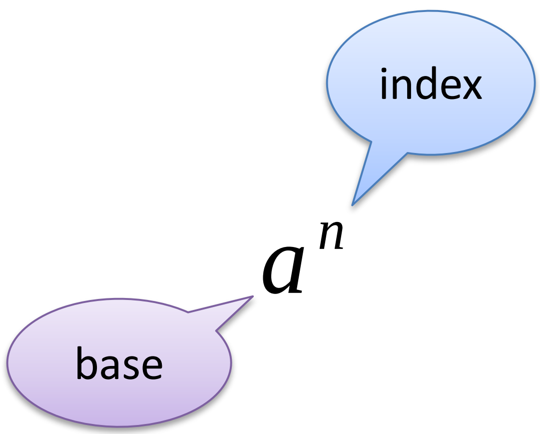
Indices

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factors}}$$



Rules of indices

1. $a^m \times a^n = a^{m+n}$

2. $a^m \div a^n = a^{m-n}$

3. $(a^m)^n = a^{m \times n}$

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Indices

Zero index

$$a^0 = 1, a \neq 0$$

Negative index

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

Rational index

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$$

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Example

Evaluate each of the following without using calculator.

(a) $27^{\frac{2}{3}}$

(b) $\dfrac{5^5 \times 25^3}{125^4}$

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Solution

(a) $$27^{\frac{2}{3}} = \left(3^3\right)^{\frac{2}{3}}$$

$$= 3^{3\times\frac{2}{3}}$$

$$= 3^2$$

$$= 9$$

(b) $$\frac{5^5 \times 25^3}{125^4} = \frac{5^5 \times \left(5^2\right)^3}{\left(5^3\right)^4}$$

$$= \frac{5^5 \times 5^6}{5^{12}}$$

$$= 5^{5+6-12}$$

$$= 5^{-1}$$

$$= \frac{1}{5}$$

Surds

Conjugate surds

$\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are conjugate surds.

Product of a pair of conjugate surds is a rational number

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2=a-b\in Q$$

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Surds

Operations on surds

1. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

2. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

3. $\sqrt{a} \times \sqrt{a} = \left(\sqrt{a}\right)^2 = a$

4. $m \times n\sqrt{a} = mn\sqrt{a}$

5. $m\sqrt{a} + n\sqrt{a} = (m+n)\sqrt{a}$

6. $m\sqrt{a} - n\sqrt{a} = (m-n)\sqrt{a}$

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Surds

Rationalising denominators

Division by surds of the form $\frac{b}{\sqrt{a}}$ can be simplified by multiplying it with $\frac{\sqrt{a}}{\sqrt{a}}$ (writing 1 as $\frac{\sqrt{a}}{\sqrt{a}}$).

- Also known as process of eliminating the surd in the denominator of a fraction.

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Example

(1) Simplify

(a) $\sqrt{20}$

(b) $13\times 2\sqrt{2}$

(2) Express $\sqrt{75}+4\sqrt{3}$ in the form of $a\sqrt{b}$.

(3) Simplify $\left(7\sqrt{2}+11\sqrt{3}\right)+\left(3\sqrt{2}-5\sqrt{3}\right)$.

(4) Express $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ with a rational denominator.

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Solution

(1) (a) $\sqrt{20} = \sqrt{4\times 5}$

$= \sqrt{4}\times\sqrt{5}$

$= 2\times\sqrt{5}$

$= 2\sqrt{5}$

(b) $13\times 2\sqrt{2} = (13\times 2)\sqrt{2}$

$= 26\sqrt{2}$

(2) $\sqrt{75} + 4\sqrt{3}$

$= \sqrt{25\times 3} + 4\sqrt{3}$

$= \sqrt{25}\times\sqrt{3} + 4\sqrt{3}$

$= 5\sqrt{3} + 4\sqrt{3}$

$= (5+4)\sqrt{3}$

$= 9\sqrt{3}$

Solution

(3) $\left(7\sqrt{2}+11\sqrt{3}\right)+\left(3\sqrt{2}-5\sqrt{3}\right)$

$$=7\sqrt{2}+11\sqrt{3}+3\sqrt{2}-5\sqrt{3}$$

$$=\left(7\sqrt{2}+3\sqrt{2}\right)+\left(11\sqrt{3}-5\sqrt{3}\right)$$

$$=10\sqrt{2}+6\sqrt{3}$$

Solution

(4) $$\frac{\sqrt{3}+1}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{\left(\sqrt{3}\right)^2+2\sqrt{3}+1}{\left(\sqrt{3}\right)^2-1^2}$$

$$=\frac{3+2\sqrt{3}+1}{3-1}$$

$$=\frac{4+2\sqrt{3}}{2}$$

$$=2+\sqrt{3}$$

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Logarithms

Index form

$$b = a^x$$

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Logarithmic form

$$\log_a b = x$$

Important results

1. $a^{\log_a b} = b$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
5. $\log 10 = 1$
6. $\ln e = 1$

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Logarithms

Laws of logarithms

1. $\log_a MN = \log_a M + \log_a N$
2. $\log_a \frac{M}{N} = \log_a M - \log_a N$
3. $\log_a M^N = N \log_a M$

Learning Tip: If $\log x = \log y$, then $x = y$.

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Example

(1) Express $\log xyz^2$ in terms of $\log x$, $\log y$ and $\log z$.

(2) Evaluate $\log_2 128$.

(3) Express $\log_a 3xy - 5\log_a y + 2$ as a single logarithm.

(4) Calculate the value of $\log_4 30$.

Solution

(1) $\log xyz^2 = \log x + \log y + \log z^2$

$= \log x + \log y + 2\log z$

(2) $\log_2 128 = \log_2 2^7$

$= 7\log_2 2$

$= 7 \times 1$

$= 7$

Solution

(3) $\log_a 3xy - 5\log_a y + 2$

$$= \log_a 3xy - \log_a y^5 + \log_a a^2$$

$$= \log_a \frac{3xy \times a^2}{y^5}$$

$$= \log_a \frac{3xa^2}{y^4}$$

(4) $\log_4 30 = \dfrac{\log_{10} 30}{\log_{10} 4} = \dfrac{1.4771}{0.6021} = 2.453$

Self-check

(1) By using the rules of indices, evaluate

$$\left(\sqrt{3}\right)^3 \times 27^{\frac{1}{4}} \times 3^{-\frac{1}{4}} .$$

(2) Simplify

(a) $\sqrt{\dfrac{49}{100}}$

(b) $5\sqrt{3} \times 8\sqrt{3}$

(3) Express $25\sqrt{7} - 6\sqrt{63}$ in the form of $a\sqrt{b}$.

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Bloom: Applying

Self-check

(4) Simplify $\left(18\sqrt{10}-6\sqrt{7}\right)-\left(9\sqrt{10}+13\sqrt{7}\right)$.

(5) Express $\dfrac{8\sqrt{5}-\sqrt{2}}{\sqrt{5}+2\sqrt{2}}$ with a rational denominator.

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Bloom: Applying

Self-check

(6) Express $\log\sqrt{\frac{y}{xz}}$ in term of $\log x$, $\log y$ and $\log z$.

(7) Evaluate $\log_6 48 - 2\log_6 2 + \log_6 3$.

(8) If $\frac{1}{2}\log_2 p = 3 - \log_2 q$, show that $pq^2 = 64$.

(9) Calculate the of $\log_{81} 3$ without using calculator.

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Bloom: Applying

Answer Self-check

(1) 9

(2) (a) $\frac{7}{10}$ (b) 120

(3) $7\sqrt{7}$

(4) $9\sqrt{10}-19\sqrt{7}$

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Bloom: Applying

Answer Self-check

(5) $\dfrac{17\sqrt{10}-44}{3}$

(6) $\dfrac{1}{2}\left(\log y - \log x - \log z\right)$

(7) 2

(9) $\dfrac{1}{4}$

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Bloom: Applying

Summary

Indices, Surd and Logarithms

Law of Logarithms

$$\log_a xy = \log_a x + \log_a y$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
$$\log_a x^b = b\log_a x$$
$$\log_a a = 1$$
$$a^{\log_a x} = x$$
$$\log_a x = \frac{\log_b x}{\log_b a}$$

Surd

$$\sqrt{p} \cdot \sqrt{q} = \sqrt{pq}$$
$$\frac{\sqrt{p}}{\sqrt{q}} = \sqrt{\frac{p}{q}}$$
$$p\sqrt{r} \pm q\sqrt{r} = (p \pm q)\sqrt{r}$$

Law of Indices

$$x^m \cdot x^n = x^{m+n}$$
$$x^m \div x^n = x^{m-n}$$
$$(x^m)^n = x^{mn}$$
$$x^0 = 1$$
$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$
$$x^{-n} = \frac{1}{x^n}$$
$$x^{\frac{1}{n}} = \sqrt[n]{x}$$
$$x^{\frac{m}{n}} = (\sqrt[n]{x})^m$$

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Bloom: Remembering

Key Terms

- Indeces
- Zero index
- Negative index
- Rational index
- Surd
- Rationalising denominators
- Conjugate surds
- Logarithms