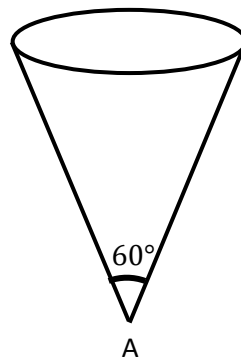


QS 015/2
Matriculation Programme
Examination
Semester I
Session 2016/2017

1. Express $\frac{x^2}{x^2-2x-3}$ in partial fractions form.
2. Evaluate the following limits, if exist.
 - a. $\lim_{x \rightarrow 2} \frac{x-2}{x^4-16}$
 - b. $\lim_{x \rightarrow \infty} \frac{(2-x)(x-1)}{(x-3)^2}$
3. Show that $\frac{\sin^2 x}{1-\cos x} = 1 + \cos x$. Hence, solve $\frac{\sin^2 x}{1-\cos x} = \cos 2x$ for $0^\circ \leq x \leq 360^\circ$.
4. Consider a function $f(x) = \frac{1}{2-\sqrt{x}}$.
 - a. Find $\lim_{x \rightarrow \infty} f(x)$ and state the equation of horizontal asymptote for f .
 - b. By using the first principle of derivative, find $f'(x)$.
5. (a) Use the derivative to find the maximum area of a rectangle that can be inscribed in a semicircle of radius 10cm.
 - (b) A cone-shaped tank as shown below.



Water flows through a hole A at rate of 6 cm^3 per second. Find the rate of change in height of the water when the volume of water in the cone is $24\pi \text{ cm}^3$

6. (a) Polynomial $P(x)$ has a remainder 3 when divided by $(x + 3)$. Find the remainder of $P(x) + 2$ when divided by $(x + 3)$.
 - (b) Polynomial $P_1(x) = x^3 + ax^2 - 5bx - 7$ has a factor $(x - 1)$ and remainder R_1 when divided by $(x + 1)$, while a polynomial $P_2(x) = x^3 - ax^2 + bx + 6$ has a remainder R_2 when divided by $(x - 1)$. Find the value of the constants a and b if $R_1 + R_2 = 5$. Hence, obtain the zeroes for $P_1(x)$.

7. Consider a function $f(x) = \sqrt{3} \cos 2x + 2 \sin 2x$.
- Express f in the form of $R \cos(2x - \alpha)$ for $R > 0$, $0^\circ \leq \alpha \leq 90^\circ$ and α to the nearest minute. State the maximum and minimum values of f .
 - Hence, solve $\sqrt{3} \cos 2x + 2 \sin 2x = -\sqrt{2}$ for $0^\circ \leq x \leq 180^\circ$. Give your answer to the nearest minute.

8. The parametric equations of a curve is given by

$$x = e^{2t+1}, \quad y = e^{-(2t-1)}$$

- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = 1$.
 - Given $z = x^2 - xy$. Express z in terms of t and find $\frac{dz}{dt}$. Hence, deduce the set value of t such that $\frac{dz}{dt}$ is positive.
9. (a) Given $f(x) = \frac{2|x|}{x} + 5x$. Compute $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$. Is f continuous at $x=0$? Give your reason.

- (b) The continuous function g is defined by

$$f(x) = \begin{cases} \sqrt{5-x}, & x < a \\ 3x-1, & x \geq a \end{cases}$$

Find the value of a .

10. By writing $\tan x$ in terms of $\sin x$ and $\cos x$, show that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

- If $y = \tan x$, find $\frac{d^2y}{dx^2}$ in terms of y . Hence, determine the range of value of x such that $\frac{d^2y}{dx^2} > 0$ for $0 < x < \pi$.
- If $y = \tan(x+y)$, find $\frac{dy}{dx}$ in terms of x and y .

Hence, show that $\frac{dy}{dx} = -\operatorname{cosec}^2 2\alpha$ when $x = y = \alpha$.

END OF QUESTION PAPER

1. Express $\frac{x^2}{x^2-2x-3}$ in partial fractions form.

SOLUTION

$$\frac{x^2}{x^2 - 2x - 3}$$

Improper Fraction

$$\begin{array}{r} 1 \\ x^2 - 2x - 3 \overline{) x^2} \\ \underline{x^2 - 2x - 3} \\ 2x + 3 \end{array}$$

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

$$\frac{x^2}{x^2 - 2x - 3} = 1 + \frac{2x + 3}{x^2 - 2x - 3}$$

$$\frac{2x + 3}{x^2 - 2x - 3} = \frac{2x + 3}{(x - 3)(x + 1)} = \frac{A}{x - 3} + \frac{B}{x + 1}$$

$$\frac{2x + 3}{(x - 3)(x + 1)} = \frac{A(x + 1) + B(x - 3)}{(x - 3)(x + 1)}$$

$$2x + 3 = A(x + 1) + B(x - 3)$$

When $x = -1$

$$2(-1) + 3 = A((-1) + 1) + B((-1) - 3)$$

$$1 = -4B$$

$$B = -\frac{1}{4}$$

When $x = 3$

$$2(3) + 3 = A((3) + 1) + B((3) - 3)$$

$$9 = 4A$$

$$A = \frac{9}{4}$$

$$\frac{2x + 3}{x^2 - 2x - 3} = \frac{9}{4(x - 3)} - \frac{1}{4(x + 1)}$$

$$\begin{aligned}\frac{x^2}{x^2 - 2x - 3} &= 1 + \frac{2x + 3}{x^2 - 2x - 3} \\ &= 1 + \frac{9}{4(x - 3)} - \frac{1}{4(x + 1)}\end{aligned}$$

2. Evaluate the following limits, if exist.

a. $\lim_{x \rightarrow 2} \frac{x-2}{x^4-16}$

b. $\lim_{x \rightarrow \infty} \frac{(2-x)(x-1)}{(x-3)^2}$

SOLUTION

a. $\lim_{x \rightarrow 2} \frac{x-2}{x^4-16} = \lim_{x \rightarrow 2} \frac{x-2}{(x^2+4)(x^2-4)}$

$$= \lim_{x \rightarrow 2} \frac{x-2}{(x^2+4)(x+2)(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{1}{(x^2+4)(x+2)}$$

$$= \frac{1}{(2^2+4)(2+2)}$$

$$= \frac{1}{32}$$

b. $\lim_{x \rightarrow \infty} \frac{(2-x)(x-1)}{(x-3)^2} = \lim_{x \rightarrow \infty} \frac{2x-2-x^2+x}{x^2-6x+9}$

$$= \lim_{x \rightarrow \infty} \frac{-x^2+3x-2}{x^2-6x+9}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{-x^2+3x-2}{x^2}}{\frac{x^2-6x+9}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{-1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{6}{x} + \frac{9}{x^2}}$$

$$= \frac{-1+0+0}{1}$$

$$= -1$$

3. Show that $\frac{\sin^2 x}{1-\cos x} = 1 + \cos x$. Hence, solve $\frac{\sin^2 x}{1-\cos x} = \cos 2x$ for $0^\circ \leq x \leq 360^\circ$.

SOLUTION

$$\begin{aligned}\frac{\sin^2 x}{1-\cos x} &= \frac{1-\cos^2 x}{1-\cos x} \\ &= \frac{(1-\cos x)(1+\cos x)}{1-\cos x} \\ &= 1 + \cos x\end{aligned}$$

$$\frac{\sin^2 x}{1-\cos x} = \cos 2x$$

$$1 + \cos x = 2 \cos^2 x - 1$$

$$2 \cos^2 x - \cos x - 2 = 0$$

Let $y = \cos x$

$$2y^2 - y - 2 = 0$$

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-2)}}{2(2)}$$

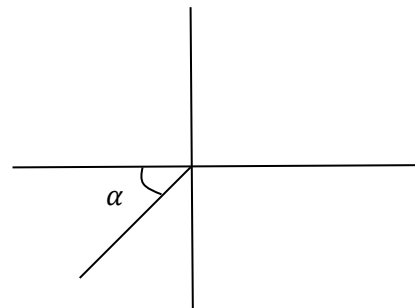
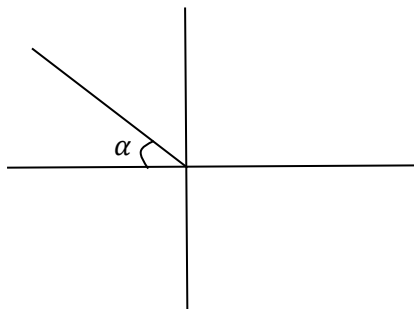
$$= \frac{1 \pm \sqrt{17}}{4}$$

$$= 1.2808, -0.7808$$

$$\cos x = 1.2808, -0.7808$$

Since $-1 \leq \cos x \leq 1$

$$\cos x = -0.7808$$



$$\begin{aligned}ax^2 + bx + c &= 0 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

$$\begin{aligned}\alpha &= \cos^{-1}0.7808 \\ &= 38.67^\circ\end{aligned}$$

Given that $0^\circ \leq x \leq 360^\circ$

$$\begin{aligned}x &= 180^\circ - 38.67, 180^\circ + 38.67 \\ &= 141.33^\circ, 218.67^\circ\end{aligned}$$

4. Consider a function $f(x) = \frac{1}{2-\sqrt{x}}$.
- Find $\lim_{x \rightarrow \infty} f(x)$ and state the equation of horizontal asymptote for f .
 - By using the first principal of derivative, find $f'(x)$.

SOLUTION

$$f(x) = \frac{1}{2-\sqrt{x}}$$

$$\begin{aligned} \text{a. } \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{1}{2-\sqrt{x}} \\ &= 0 \end{aligned}$$

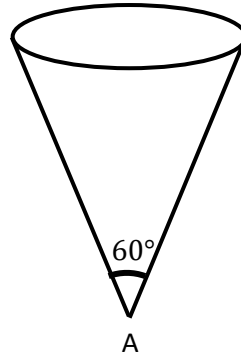
$\therefore f(x) = 0$ is horizontal asymptote.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} \text{b. } f'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} [f(x+h) - f(x)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{2-\sqrt{x+h}} - \frac{1}{2-\sqrt{x}} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(2-\sqrt{x}) - (2-\sqrt{x+h})}{(2-\sqrt{x+h})(2-\sqrt{x})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2-\sqrt{x} - 2 + \sqrt{x+h}}{(2-\sqrt{x+h})(2-\sqrt{x})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-\sqrt{x} + \sqrt{x+h}}{(2-\sqrt{x+h})(2-\sqrt{x})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sqrt{x+h} - \sqrt{x}}{(2-\sqrt{x+h})(2-\sqrt{x})} \right] \left[\frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x+h) - (\sqrt{x}\sqrt{x+h}) + (\sqrt{x}\sqrt{x+h}) - (x)}{(2-\sqrt{x+h})(2-\sqrt{x})(\sqrt{x+h} + \sqrt{x})} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h}{(2-\sqrt{x+h})(2-\sqrt{x})(\sqrt{x+h} + \sqrt{x})} \right] \\ &= \lim_{h \rightarrow 0} \left[\frac{1}{(2-\sqrt{x+h})(2-\sqrt{x})(\sqrt{x+h} + \sqrt{x})} \right] \end{aligned}$$

$$= \frac{1}{(2 - \sqrt{x})(2 - \sqrt{x})(\sqrt{x} + \sqrt{x})}$$
$$= \frac{1}{2\sqrt{x}(2 - \sqrt{x})^2}$$

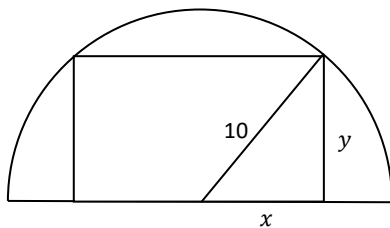
5. (a) Use the derivative to find the maximum area of a rectangle that can be inscribed in a semicircle of radius 10cm.
- (b) A cone-shaped tank as shown below.



Water flows through a hole A at rate of 6 cm^3 per second. Find the rate of change in height of the water when the volume of water in the cone is $24\pi \text{ cm}^3$

SOLUTION

(a)



$$x^2 + y^2 = 100$$

$$y^2 = 100 - x^2$$

$$y = (100 - x^2)^{\frac{1}{2}}$$

Area of rectangle

$$A = 2xy$$

$$A = 2x(100 - x^2)^{\frac{1}{2}}$$

$$u = 2x$$

$$u' = 2$$

$$v = (100 - x^2)^{\frac{1}{2}}$$

$$v' = \frac{1}{2}(100 - x^2)^{-\frac{1}{2}} \frac{d}{dx}(100 - x^2)$$

$$= \frac{1}{2(100 - x^2)^{\frac{1}{2}}}(-2x)$$

$$= \frac{-2x}{2(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{-x}{(100 - x^2)^{\frac{1}{2}}}$$

$$\frac{dA}{dx} = uv' + vu'$$

$$= (2x) \left(\frac{-x}{(100 - x^2)^{\frac{1}{2}}} \right) + \left((100 - x^2)^{\frac{1}{2}} \right) (2)$$

$$= \frac{-2x^2}{(100 - x^2)^{\frac{1}{2}}} + 2(100 - x^2)^{\frac{1}{2}}$$

$$= \frac{-2x^2 + 2(100 - x^2)^{\frac{1}{2}}(100 - x^2)^{\frac{1}{2}}}{(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{-2x^2 + 2(100 - x^2)}{(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{-2x^2 + 200 - 2x^2}{(100 - x^2)^{\frac{1}{2}}}$$

$$= \frac{200 - 4x^2}{(100 - x^2)^{\frac{1}{2}}}$$

$$\text{Let } \frac{dA}{dx} = 0$$

$$\frac{200 - 4x^2}{(100 - x^2)^{\frac{1}{2}}} = 0$$

$$200 - 4x^2 = 0$$

$$4x^2 = 200$$

$$x^2 = 50$$

$$x = \pm\sqrt{50}$$

Since $x \geq 0$

$$x = \sqrt{50}$$

$$\frac{dA}{dx} = \frac{-4x^2 + 200}{(100 - x^2)^{\frac{1}{2}}}$$

$$u = -4x^2 + 200$$

$$v = (100 - x^2)^{\frac{1}{2}}$$

$$u' = -8x$$

$$v' = \frac{1}{2}(100 - x^2)^{-\frac{1}{2}} \frac{d}{dx}(100 - x^2)$$

$$= \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{(100 - x^2)^{\frac{1}{2}}}$$

$$\frac{d^2A}{dx^2} = \frac{vu' - uv'}{v^2}$$

$$= \frac{(100 - x^2)^{\frac{1}{2}}(-8x) - (-4x^2 + 200)\left(\frac{-x}{(100 - x^2)^{\frac{1}{2}}}\right)}{\left((100 - x^2)^{\frac{1}{2}}\right)^2}$$

$$= \frac{-8x(100 - x^2)^{\frac{1}{2}} + \left[\frac{x(-4x^2 + 200)}{(100 - x^2)^{\frac{1}{2}}}\right]}{\left((100 - x^2)^{\frac{1}{2}}\right)^2}$$

$$\begin{aligned}
&= \frac{\left[\frac{-8x(100-x^2)^{\frac{1}{2}}(100-x^2)^{\frac{1}{2}} + (-4x^3 + 200x)}{(100-x^2)^{\frac{1}{2}}} \right]}{100-x^2} \\
&= \left[\frac{-8x(100-x^2) + (-4x^3 + 200x)}{(100-x^2)^{\frac{1}{2}}} \right] \left[\frac{1}{100-x^2} \right] \\
&= \frac{-800x + 8x^3 - 4x^3 + 200x}{(100-x^2)^{\frac{3}{2}}} \\
&= \frac{4x^3 - 600x}{(100-x^2)^{\frac{3}{2}}}
\end{aligned}$$

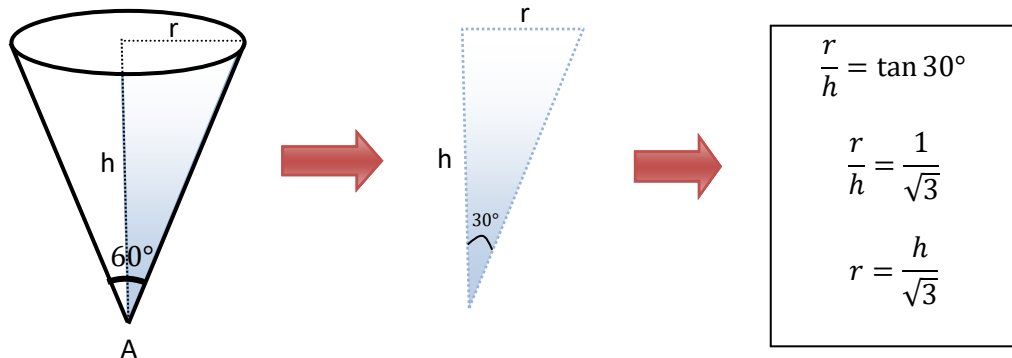
When $x = \sqrt{50}$

$$\frac{d^2A}{dx^2} = \frac{4(\sqrt{50})^3 - 600\sqrt{50}}{\left[100 - (\sqrt{50})^2\right]^{\frac{3}{2}}} = -8 < 0 \text{ (max)}$$

The maximum area, $A_{max} = 2x(100-x^2)^{\frac{1}{2}}$

$$\begin{aligned}
&= 2\sqrt{50} \left(100 - \sqrt{50}^2\right)^{\frac{1}{2}} \\
&= 2\sqrt{50}(50)^{\frac{1}{2}} \\
&= 2\sqrt{50}\sqrt{50} \\
&= 2(50) \\
&= 100\text{cm}^2
\end{aligned}$$

(b)



$$\frac{dv}{dt} = -6\text{cm}^3\text{s}^{-1}$$

Find $\frac{dh}{dt}$ when $v = 24\pi\text{cm}^3$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$v = \frac{1}{3}\pi r^2 h$$

Since $r = \frac{h}{\sqrt{3}}$

$$v = \frac{1}{3}\pi \left(\frac{h}{\sqrt{3}}\right)^2 h$$

$$= \frac{1}{3}\pi \frac{h^3}{3}$$

$$= \frac{\pi}{9}h^3$$

$$\frac{dv}{dh} = \frac{\pi h^2}{3}$$

$$\frac{dh}{dv} = \frac{3}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{3}{\pi h^2} \cdot (-6)$$

$$\frac{dh}{dt} = -\frac{18}{\pi h^2}$$

when $v = 24\pi$

$$\frac{\pi}{9}h^3 = 24\pi$$

$$h^3 = 24\pi \times \frac{9}{\pi}$$

$$= 216$$

$$h = 6$$

$$\frac{dh}{dt} = -\frac{18}{\pi 6^2}$$

$$\frac{dh}{dt} = -\frac{18}{\pi(6)^2}$$

$$= -\frac{1}{2\pi} \text{ cms}^{-1}$$

6. (a) Polynomial $P(x)$ has a remainder 3 when divided by $(x + 3)$. Find the remainder of $P(x) + 2$ when divided by $(x + 3)$.
- (b) Polynomial $P_1(x) = x^3 + ax^2 - 5bx - 7$ has a factor $(x - 1)$ and remainder R_1 when divided by $(x + 1)$, while a polynomial $P_2(x) = x^3 - ax^2 + bx + 6$ has a remainder R_2 when divided by $(x - 1)$. Find the value of the constants a and b if $R_1 + R_2 = 5$. Hence, obtain the zeroes for $P_1(x)$.

SOLUTION

(a) $P(-3) = 3$

$$P(x) = Q(x)D(x) + R(x)$$

$$P(x) = Q(x)(x + 3) + 3$$

$$P(x) + 2 = Q(x)(x + 3) + 3 + 2$$

$$P(x) + 2 = Q(x)(x + 3) + 5$$

$$\therefore R(x) = 5 \text{ when } P(x) + 2 \text{ divided by } (x + 3)$$

(b) $P_1(x) = x^3 + ax^2 - 5bx - 7$

$$P_1(1) = 0$$

$$(1)^3 + a(1)^2 - 5b(1) - 7 = 0$$

$$1 + a - 5b - 7 = 0$$

$$a - 5b = 6 \quad \dots\dots\dots (1)$$

$$P_1(-1) = R_1$$

$$(-1)^3 + a(-1)^2 - 5b(-1) - 7 = R_1$$

$$-1 + a + 5b - 7 = R_1$$

$$a + 5b = R_1 + 8$$

$$R_1 = a + 5b - 8 \quad \dots\dots\dots (2)$$

$$P_2(x) = x^3 - ax^2 + bx + 6$$

$$P_2(1) = R_2$$

$$(1)^3 - a(1)^2 + b(1) + 6 = R_2$$

$$1 - a + b + 6 = R_2$$

$$-a + b = R_2 - 7$$

$$R_2 = -a + b + 7 \quad \dots\dots\dots (3)$$

$$(2) + (3)-$$

$$R_1 + R_2 = 6b - 1$$

$$\text{Given that } R_1 + R_2 = 5$$

$$5 = 6b - 1$$

$$6b = 6$$

$$b = 1 \quad \dots\dots\dots (4)$$

Substitute (4) into (1)

$$a - 5(1) = 6$$

$$a = 11$$

$$\therefore a = 11, b = 1$$

$$\begin{aligned} P_1(x) &= x^3 + 11x^2 - 5x - 7 \\ &= (x - 1)Q(x) \end{aligned}$$

$P_1(x) = x^3 + ax^2 - 5bx - 7$ has a factor of $(x - 1)$

$$\begin{array}{r}
 x^2 + 12x + 7 \\
 x-1 \overline{) x^3 + 11x^2 - 5x - 7} \\
 \underline{x^3 - x^2} \\
 12x^2 - 5x - 7 \\
 \underline{12x^2 - 12x} \\
 7x - 7 \\
 \underline{7x - 7} \\
 0
 \end{array}$$

$$\begin{aligned}
 P_1(x) &= x^3 + 11x^2 - 5x - 7 \\
 &= (x - 1)(x^2 + 12x + 7)
 \end{aligned}$$

To find the zeroes of $P_1(x)$

$$P_1(x) = 0$$

$$(x - 1)(x^2 + 12x + 7) = 0$$

$$x - 1 = 0$$

$$x = 1$$

$$(x^2 + 12x + 7) = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-12 \pm \sqrt{144 - 28}}{2}$$

$$= \frac{-12 \pm \sqrt{116}}{2}$$

$$= -6 \pm \sqrt{29}$$

Hence, the zeroes of $P_1(x)$ are $1, -6 \pm \sqrt{29}$

7. Consider a function $f(x) = \sqrt{3} \cos 2x + 2 \sin 2x$.
- Express f in the form of $R \cos(2x - \alpha)$ for $R > 0$, $0^\circ \leq \alpha \leq 90^\circ$ and α to the nearest minute. State the maximum and minimum values of f .
 - Hence, solve $\sqrt{3} \cos 2x + 2 \sin 2x = -\sqrt{2}$ for $0^\circ \leq x \leq 180^\circ$. Give your answer to the nearest minute.

SOLUTION

(a)

$$f(x) = \sqrt{3} \cos 2x + 2 \sin 2x$$

$$\sqrt{3} \cos 2x + 2 \sin 2x = R \cos(2x - \alpha)$$

$$\sqrt{3} \cos 2x + 2 \sin 2x = R[\cos 2x \cos \alpha + \sin 2x \sin \alpha]$$

$$\sqrt{3} \cos 2x + 2 \sin 2x = R \cos 2x \cos \alpha + R \sin 2x \sin \alpha$$

$$R \cos \alpha = \sqrt{3} \quad \dots\dots\dots (1)$$

$$R \sin \alpha = 2 \quad \dots\dots\dots (2)$$

$$(1)^2 + (2)^2$$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = (\sqrt{3})^2 + (2)^2$$

$$R^2(\cos^2 \alpha + \sin^2 \alpha) = 3 + 4$$

$$R^2 = 7$$

$$R = \sqrt{7}$$

$$(2) \div (1)$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{2}{\sqrt{3}}$$

$$\tan \alpha = \frac{2}{\sqrt{3}}$$

$$\alpha = 49.11^\circ$$

$$\therefore f(x) = \sqrt{7} \cos(2x - 49.11^\circ)$$

$$\text{Since } -1 \leq \cos(2x - 49.11^\circ) \leq 1$$

$$\begin{aligned} \cos(A - B) &= \cos A \cos B + \sin A \sin B \\ \cos(2x - \alpha) &= \cos 2x \cos \alpha + \sin 2x \sin \alpha \end{aligned}$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$-\sqrt{7} \leq \sqrt{7} \cos(2x - 49.11^\circ) \leq \sqrt{7}$$

$$\therefore \text{Maximum value of } f = \sqrt{7}$$

$$\therefore \text{Minimum value of } f = -\sqrt{7}$$

(b)

$$\sqrt{3} \cos 2x + 2 \sin 2x = -\sqrt{2} \text{ for } 0^\circ \leq x \leq 180^\circ$$

$$\sqrt{7} \cos(2x - 49.11^\circ) = -\sqrt{2}$$

$$\cos(2x - 49.11^\circ) = -\frac{\sqrt{2}}{\sqrt{7}}$$

$$2x - 49.11^\circ = 180^\circ - 57.69^\circ, 180^\circ + 57.69^\circ$$

$$2x - 49.11^\circ = 122.31^\circ, \quad 237.69^\circ$$

$$2x = 171.12^\circ, \quad 286.80^\circ$$

$$x = 85.56^\circ, \quad 143.4^\circ$$

8. The parametric equations of a curve is given by

$$x = e^{2t+1}, \quad y = e^{-(2t-1)}$$

(c) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = 1$.

(d) Given $z = x^2 - xy$. Express z in terms of t and find $\frac{dz}{dt}$. Hence, deduce the set value of t such that $\frac{dz}{dt}$ is positive.

SOLUTION

$$x = e^{2t+1}, \quad y = e^{-(2t-1)}$$

$$\frac{dx}{dt} = 2e^{2t+1} \quad \frac{dy}{dt} = -2e^{-(2t-1)}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= -2e^{-(2t-1)} \cdot \frac{1}{2e^{2t+1}}$$

$$= \frac{-2e^{-(2t-1)}}{2e^{2t+1}}$$

$$= \frac{-e^{-(2t-1)}}{e^{2t+1}}$$

$$= -\frac{1}{e^{4t}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

When $t = 1$

$$\frac{dy}{dx} = -\frac{1}{e^{4(1)}}$$

$$= -\frac{1}{e^4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left[\frac{dy}{dx} \right] \cdot \frac{dt}{dx}$$

$$= \left[\frac{d}{dt} \left(-\frac{1}{e^{4t}} \right) \right] \cdot \left[\frac{1}{2e^{2t+1}} \right]$$

$$= \left[\frac{d}{dt} (-e^{-4t}) \right] \cdot \left[\frac{1}{2e^{2t+1}} \right]$$

$$\begin{aligned}
 &= 4e^{-4t} \cdot \left[\frac{1}{2e^{2t+1}} \right] \\
 &= \frac{4e^{-4t}}{2e^{2t+1}} \\
 &= \frac{2e^{-4t}}{e^{2t+1}} \\
 &= \frac{2}{e^{6t+1}}
 \end{aligned}$$

When $t = 1$

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{2}{e^{6t+1}} \\
 &= \frac{2}{e^{6(1)+1}} \\
 &= \frac{2}{e^7}
 \end{aligned}$$

$$\begin{aligned}
 z &= x^2 - xy \\
 &= (e^{2t+1})^2 - (e^{2t+1})(e^{-(2t-1)}) \\
 &= e^{2(2t+1)} - e^{2t+1-2t+1} \\
 &= e^{4t+2} - e^2
 \end{aligned}$$

$$(a^m)^n = a^{mn}$$

$$a^m + a^n = a^{m+n}$$

$$\frac{dz}{dt} = 4e^{4t+2}$$

For $\frac{dz}{dt} > 0$

$$4e^{4t+2} > 0$$

$$e^{4t+2} > 0$$

Solution set: $\{t: t \in \mathcal{R}\}$

9. (a) Given $f(x) = \frac{2|x|}{x} + 5x$. Compute $\lim_{x \rightarrow 0^+} f(x)$ and $\lim_{x \rightarrow 0^-} f(x)$. Is f continuous at $x=0$? Give your reason.

(b) The continuous function g is defined by

$$f(x) = \begin{cases} \sqrt{5-x}, & x < a \\ 3x-1, & x \geq a \end{cases}$$

Find the value of a .

SOLUTION

(a) $f(x) = \frac{2|x|}{x} + 5x$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{2x}{x} + 5x, & x > 0 \\ \frac{2(-x)}{x} + 5x, & x < 0 \end{cases}$$

$$f(x) = \begin{cases} 2 + 5x, & x > 0 \\ -2 + 5x, & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2 + 5x$$

$$= 2 + 5(0)$$

$$= 2$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -2 + 5x$$

$$= -2 + 5(0)$$

$$= -2$$

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

$\lim_{x \rightarrow 0} f(x)$ does not exist

Since $\lim_{x \rightarrow 0} f(x)$ does not exist, therefore f is not continuous at $x = 0$

$$(b) f(x) = \begin{cases} \sqrt{5-x}, & x < a \\ 3x-1, & x \geq a \end{cases}$$

$$f(a) = 3(a) - 1 = 3a - 1$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} 3x - 1 = 3(a) - 1 = 3a - 1$$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} \sqrt{5-x} = \sqrt{5-a}$$

$$\text{Since } g \text{ is continuous at } x = a \rightarrow \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$

$$3a - 1 = \sqrt{5-a}$$

$$(3a - 1)^2 = (\sqrt{5-a})^2$$

$$9a^2 - 6a + 1 = 5 - a$$

$$9a^2 - 5a - 4 = 0$$

$$(9a + 4)(a - 1) = 0$$

$$(9a + 4) = 0 \quad (a - 1) = 0$$

$$a = -\frac{4}{9} \quad a = 1$$

$$\text{When } a = -\frac{4}{9},$$

$$3\left(-\frac{4}{9}\right) - 1 \neq \sqrt{5 - \left(-\frac{4}{9}\right)}$$

$$\therefore a \neq -\frac{4}{9}$$

$$\text{When } a = 1,$$

$$3(1) - 1 = \sqrt{5 - (1)}$$

$$\therefore a = 1$$

g is continuous at $x = a$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$

$$3a - 1 = \sqrt{5-a}$$

10. By writing $\tan x$ in terms of $\sin x$ and $\cos x$, show that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

- a. If $y = \tan x$, find $\frac{d^2y}{dx^2}$ in terms of y . Hence, determine the range of value of x such that $\frac{d^2y}{dx^2} > 0$ for $0 < x < \pi$.
- b. If $y = \tan(x + y)$, find $\frac{dy}{dx}$ in terms of x and y .

Hence, show that $\frac{dy}{dx} = -\operatorname{cosec}^2 2\alpha$ when $x = y = \alpha$.

SOLUTION

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$u = \sin x$$

$$v = \cos x$$

$$u' = \cos x$$

$$v' = -\sin x$$

=

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

(a) $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

$$= 1 + \tan^2 x$$

$$= 1 + y^2$$

$$\frac{d^2y}{dx^2} = 2y \frac{dy}{dx}$$

$$= 2y(1 + y^2)$$

$$1 + \tan^2 x = \sec^2 x$$

$$\text{For } \frac{d^2y}{dx^2} > 0 \quad 0 < x < \pi$$

$$2y(1 + y^2) > 0$$

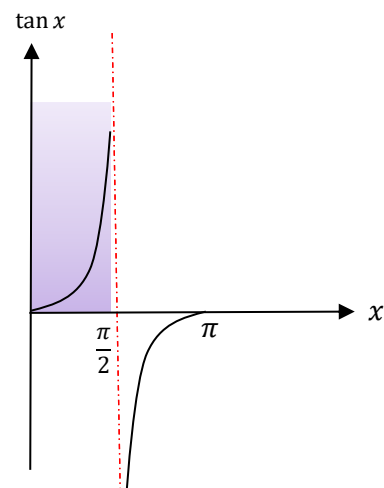
$$\text{Since } 1 + y^2 > 0$$

$$2y > 0$$

$$y > 0$$

$$\tan x > 0$$

$$\therefore \text{Solution interval is } \left(0, \frac{\pi}{2}\right)$$



(b) $y = \tan(x + y)$

$$\frac{dy}{dx} = \sec^2(x + y) \frac{d}{dx}[x + y]$$

$$\frac{dy}{dx} = \sec^2(x + y) \left[1 + \frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = \sec^2(x + y) + \sec^2(x + y) \frac{dy}{dx}$$

$$\frac{dy}{dx} - \sec^2(x + y) \frac{dy}{dx} = \sec^2(x + y)$$

$$\frac{dy}{dx} [1 - \sec^2(x + y)] = \sec^2(x + y)$$

$$\frac{dy}{dx} = \frac{\sec^2(x + y)}{1 - \sec^2(x + y)}$$

when $x = y = \alpha$

$$\frac{dy}{dx} = \frac{\sec^2(\alpha + \alpha)}{1 - \sec^2(\alpha + \alpha)}$$

$$= \frac{\sec^2 2\alpha}{1 - \sec^2 2\alpha}$$

$$= \frac{\sec^2 2\alpha}{-\tan^2 2\alpha}$$

$$= \frac{1}{\frac{\cos^2 2\alpha}{\sin^2 2\alpha} - \frac{\sin^2 2\alpha}{\cos^2 2\alpha}}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 - \sec^2 \theta = \tan^2 \theta$$

$$\begin{aligned} &= \frac{1}{\cos^2 2\alpha} \left(-\frac{\cos^2 2\alpha}{\sin^2 2\alpha} \right) \\ &= -\frac{1}{\sin^2 2\alpha} \\ &= -\operatorname{cosec}^2 2\alpha \end{aligned}$$