

QS 015/2

**Matriculation Programme
Examination**

Semester I

Session 2015/2016

1. Express $\frac{5x^2+4x+4}{(x^2-4)(x+2)}$ in the form of partial fractions.

2. Evaluate the following(if exist):
 - a) $\lim_{x \rightarrow 2} \frac{x^2+4x-12}{|x-2|}$
 - b) $\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x}$

3. Find the derivative of the following functions:
 - a) $f(x) = \cot\sqrt{4x^2 + 1}$
 - b) $f(x) = e^{2x} \ln(3x + 4)$

4. Given $\operatorname{cosec}^2 x - \cot x = 3$, show that $\cot^2 x - \cot x - 2 = 0$. Hence, solve the equation $\operatorname{cosec}^2 x - \cot x = 3$ for $0 \leq x \leq \pi$.

5. A polynomial $P(x) = 2x^4 + ax^3 + bx^2 - 17x + c$ where a, b and c are constants, has factors $(x + 2)$ and $(x - 1)$. When $P(x)$ is divided by $(x + 1)$, the remainder is 8. Find the values of a, b and c . Hence, factorize $P(x)$ completely and state its zeroes.

6. (a) Evaluate $\lim_{x \rightarrow \infty} \frac{\sqrt{2x^2+3x}}{5x+1}$.

(b) Given $f(x) = \begin{cases} 5 - px & , \quad -2 < x \leq -1 \\ x^2 + px + q & , \quad -1 < x \leq 2 \\ \frac{x^2-4}{x-2} & , \quad x > 2 \end{cases}$
 - (i) Find the values of p and q if $f(x)$ is continuous for all real values of x .
 - (ii) Sketch the graph of $f(x)$ using the values p and q obtained in part (i).

7. A curve is given by the parametric equations $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$.
- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .
 - Obtain the coordinates of the stationary points of the curve and determine the nature of the points.
8. (a) If $y^2 - 2y\sqrt{1+x^2} + x^2 = 0$, show that $\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$.
- (b) Water is running at a steady rate of $36\pi \text{ cm}^3 \text{ s}^{-1}$ into a right inverted circular cone with a semi-vertical angle of 45° .
- Find the rate of increasing in water depth when the water level is 3 cm.
 - Find the time taken when the depth of the water is 18cm.
9. (a) Determine the values of R and α , where $R > 0$ and $0^\circ < \alpha < 90^\circ$ so that $3 \sin \theta - 4 \cos \theta = R \sin(\theta - \alpha)$.
- (b) Hence, solve the equation $3 \sin \theta - 4 \cos \theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$.
- (c) From the answer obtained in part (b), find the value of θ for $0^\circ \leq \theta \leq 360^\circ$ so that $f(\theta) = \frac{1}{3 \sin \theta - 4 \cos \theta + 15}$ is minimum. Hence, find the minimum value of f .
10. (a) Find the value of k if the slope of the curve $x^3 + kx^2y - 2y^2 = 0$ at the point $(-1,1)$ is -3.
- (b) Given $y = \frac{\sin x}{1 + \cos x}$.
- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x .
 - Hence, show that $\frac{d^3y}{dx^3} - y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$.

1. Express $\frac{5x^2+4x+4}{(x^2-4)(x+2)}$ in the form of partial fractions.

SOLUTION

$$\frac{5x^2 + 4x + 4}{(x^2 - 4)(x + 2)} = \frac{5x^2 + 4x + 4}{(x - 2)(x + 2)(x + 2)}$$

$$= \frac{5x^2 + 4x + 4}{(x - 2)(x + 2)^2}$$

$$= \frac{A}{(x - 2)} + \frac{B}{(x + 2)} + \frac{C}{(x + 2)^2}$$

$$= \frac{A(x + 2)^2 + B(x + 2)(x - 2) + C(x - 2)}{(x - 2)(x + 2)^2}$$

$$\frac{5x^2 + 4x + 4}{(x - 2)(x + 2)^2} = \frac{A(x + 2)^2 + B(x + 2)(x - 2) + C(x - 2)}{(x - 2)(x + 2)^2}$$

$$5x^2 + 4x + 4 = A(x + 2)^2 + B(x + 2)(x - 2) + C(x - 2)$$

When $x = 2$:

$$5(2)^2 + 4(2) + 4 = A(2 + 2)^2$$

$$32 = 16A$$

$$A = 2$$

When $x = -2$:

$$5(-2)^2 + 4(-2) + 4 = C[(-2) - 2]$$

$$16 = -4C$$

$$C = -4$$

When $x = 0, A = 2, C = -4$:

$$5(0)^2 + 4(0) + 4 = 2[(0) + 2]^2 + B(0 + 2)(0 - 2) + (-4)(0 - 2)$$

$$4 = 2(4) - 4B + 8$$

$$4B = 12$$

$$B = 3$$

$$\frac{5x^2 + 4x + 4}{(x^2 - 4)(x + 2)} = \frac{2}{(x - 2)} + \frac{3}{(x + 2)} - \frac{4}{(x + 2)^2}$$

2. Evaluate the following(if exist):

$$\text{a) } \lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{|x - 2|}$$

$$\text{b) } \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$$

SOLUTION

(a)

$$\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{|x - 2|} = \lim_{x \rightarrow 2} \frac{(x + 6)(x - 2)}{|x - 2|}$$

$$|x - 2| = \begin{cases} (x - 2), & x \geq 2 \\ -(x - 2), & x < 2 \end{cases}$$

$$\frac{(x + 6)(x - 2)}{|x - 2|} = \begin{cases} \frac{(x + 6)(x - 2)}{x - 2}, & x \geq 2 \\ \frac{(x + 6)(x - 2)}{-(x - 2)}, & x < 2 \end{cases}$$

$$= \begin{cases} x + 6, & x \geq 2 \\ -(x + 6), & x < 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} \frac{(x + 6)(x - 2)}{|x - 2|} = \lim_{x \rightarrow 2^-} -(x + 6)$$

$$= -(2 + 6)$$

$$= -8$$

$$\lim_{x \rightarrow 2^+} \frac{(x + 6)(x - 2)}{|x - 2|} = \lim_{x \rightarrow 2^+} (x + 6)$$

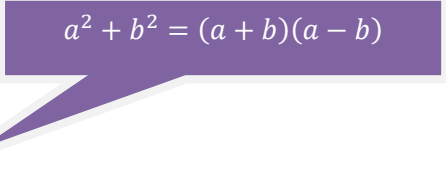
$$= (2 + 6)$$

$$= 8$$

Since $\lim_{x \rightarrow 2^-} \frac{(x+6)(x-2)}{|x-2|}$
 $\neq \lim_{x \rightarrow 2^+} \frac{(x+6)(x-2)}{|x-2|}$, therefore $\lim_{x \rightarrow 2} \frac{x^2 + 4x - 12}{|x-2|}$ does not exist.

(b)

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} &= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1^2 - (\sqrt{x})^2} \\ &= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 + \sqrt{x})(1 - \sqrt{x})} \\ &= \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} \\ &= \frac{1}{1 + \sqrt{1}} \\ &= \frac{1}{2}\end{aligned}$$


$$a^2 + b^2 = (a + b)(a - b)$$

3. Find the derivative of the following functions:

a) $f(x) = \cot\sqrt{4x^2 + 1}$

b) $f(x) = e^{2x} \ln(3x + 4)$

SOLUTION

(a)

$$f(x) = \cot\sqrt{4x^2 + 1}$$

$$f(x) = \cot(4x^2 + 1)^{\frac{1}{2}}$$

$$f'(x) = -\operatorname{cosec}^2(4x^2 + 1)^{\frac{1}{2}} \frac{d}{dx}(4x^2 + 1)^{\frac{1}{2}}$$

$$= -\operatorname{cosec}^2(4x^2 + 1)^{\frac{1}{2}} \left(\frac{1}{2}\right) (4x^2 + 1)^{-\frac{1}{2}} \frac{d}{dx}(4x^2 + 1)$$

$$= -\operatorname{cosec}^2(4x^2 + 1)^{\frac{1}{2}} \left(\frac{1}{2}\right) (4x^2 + 1)^{-\frac{1}{2}} (8x)$$

$$= -\left(\frac{1}{2}\right) (8x) (4x^2 + 1)^{-\frac{1}{2}} \operatorname{cosec}^2(4x^2 + 1)^{\frac{1}{2}}$$

$$= -\frac{(4x)}{(4x^2 + 1)^{\frac{1}{2}}} \operatorname{cosec}^2(4x^2 + 1)^{\frac{1}{2}}$$

$$= -\frac{4x \operatorname{cosec}^2\sqrt{(4x^2 + 1)}}{\sqrt{(4x^2 + 1)}}$$

(b)

$$f(x) = e^{2x} \ln(3x + 4)$$

$$u = e^{2x}$$

$$v = \ln(3x + 4)$$

$$u' = e^{2x} \frac{d}{dx}(2x)$$

$$v' = \frac{1}{3x+4} \frac{d}{dx}(3x + 4)$$

$$= 2e^{2x}$$

$$= \frac{3}{3x+4}$$

$$f'(x) = (e^{2x}) \left(\frac{3}{3x+4} \right) + [\ln(3x+4)](2e^{2x})$$

$$= \frac{3e^{2x}}{3x+4} + 2e^{2x} \ln(3x+4)$$

$$= e^{2x} \left[\frac{3}{3x+4} + 2 \ln(3x+4) \right]$$

4. Given $\operatorname{cosec}^2 x - \cot x = 3$, show that $\cot^2 x - \cot x - 2 = 0$. Hence, solve the equation $\operatorname{cosec}^2 x - \cot x = 3$ for $0 \leq x \leq \pi$.

SOLUTION

$$\operatorname{cosec}^2 x - \cot x = 3$$

$$(1 + \cot^2 x) - \cot x = 3$$

$$\cot^2 x - \cot x + 1 = 3$$

$$\cot^2 x - \cot x + 1 - 3 = 0$$

$$\cot^2 x - \cot x - 2 = 0$$

$$\operatorname{cosec}^2 x - \cot x = 3, \quad 0 \leq x \leq \pi$$

$$\cot^2 x - \cot x - 2 = 0$$

Let $u = \cot x$

$$u^2 - u - 2 = 0$$

$$(u - 2)(u + 1) = 0$$

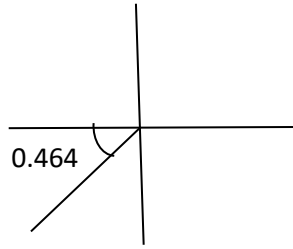
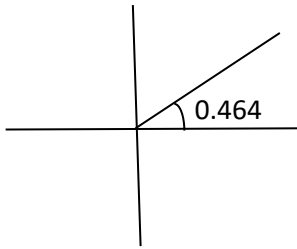
$$u = 2 \quad u = -1$$

$$\cot x = 2 \quad \text{or} \quad \cot x = -1$$

$$\frac{1}{\tan x} = 2 \quad \text{or} \quad \frac{1}{\tan x} = -1$$

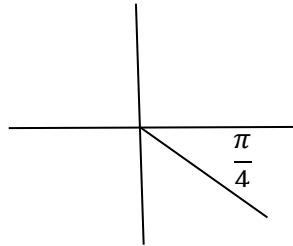
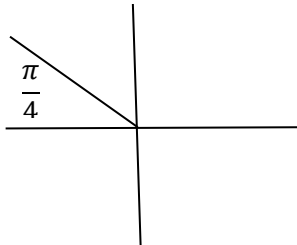
$$\tan x = \frac{1}{2} \quad \text{or} \quad \tan x = -1$$

$$\text{For } \tan x = \frac{1}{2}$$



$$x = 0.464$$

$$\text{For } \tan x = -1$$



$$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore x = 0.464, \frac{3\pi}{4}$$

5. A polynomial $P(x) = 2x^4 + ax^3 + bx^2 - 17x + c$ where a, b and c are constants, has factors $(x + 2)$ and $(x - 1)$. When $P(x)$ is divided by $(x + 1)$, the remainder is 8. Find the values of a, b and c . Hence, factorize $P(x)$ completely and state its zeroes.

SOLUTION

$$P(-2) = 0$$

$$P(1) = 0$$

$$P(-1) = 8$$

For $P(-2) = 0$

$$P(x) = 2x^4 + ax^3 + bx^2 - 17x + c$$

$$P(-2) = 2(-2)^4 + a(-2)^3 + b(-2)^2 - 17(-2) + c$$

$$0 = 32 - 8a + 4b + 34 + c$$

$$8a - 4b - c = 66 \quad \dots\dots\dots (1)$$

For $P(1) = 0$

$$P(1) = 2(1)^4 + a(1)^3 + b(1)^2 - 17(1) + c$$

$$0 = 2 + a + b - 17 + c$$

$$a + b + c = 15 \quad \dots\dots\dots (2)$$

For $P(-1) = 8$

$$P(-1) = 2(-1)^4 + a(-1)^3 + b(-1)^2 - 17(-1) + c$$

$$8 = 2 - a + b + 17 + c$$

$$a - b - c = 11 \quad \dots\dots\dots (3)$$

(2) + (3)

$$2a = 26$$

$$a = 13 \quad \dots\dots\dots (4)$$

Substitute (4) into (1)

$$8(13) - 4b - c = 66$$

$$4b + c = 38 \quad \dots\dots\dots (5)$$

Substitute (4) into (2)

$$13 + b + c = 15$$

$$b + c = 2 \quad \dots\dots\dots (6)$$

(5) - (6)

$$3b = 36$$

$$b = 12 \quad \dots\dots\dots (7)$$

Substitute (7) into (6)

$$12 + c = 2$$

$$c = -10$$

$$\therefore a = 13, \quad b = 12, \quad c = -10$$

$$P(x) = 2x^4 + 13x^3 + 12x^2 - 17x - 10$$

$$P(-2) = 0 \rightarrow (x + 2) \text{ is a factor of } P(x)$$

$$P(1) = 0 \rightarrow (x - 1) \text{ is a factor of } P(x)$$

$$P(x) = (x + 2)(x - 1)Q(x)$$

$$= (x^2 + x - 2)Q(x)$$

$$\begin{array}{r} 2x^2 + 11x + 5 \\ x^2 + x - 2 \overline{) 2x^4 + 13x^3 + 12x^2 - 17x - 10} \\ \underline{2x^4 + 2x^3 - 4x^2} \\ 11x^3 + 16x^2 - 17x - 10 \\ \underline{11x^3 + 11x^2 - 22x} \\ 5x^2 + 5x - 10 \\ \underline{5x^2 + 5x - 10} \\ 0 \end{array}$$

$$P(x) = (x + 2)(x - 1)(2x^2 + 11x + 5)$$

$$= (x + 2)(x - 1)(2x + 1)(x + 5)$$

When $P(x) = 0$

$$(x + 2)(x - 1)(2x + 1)(x + 5) = 0$$

$$x = -2, \quad x = 1, \quad x = -\frac{1}{2}, \quad x = -5$$

Factor Theorem

If $P(a) = 0$ then
 $(x - a)$ is a factor of $P(x)$

6. (a) Evaluate $\lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+3x}}{5x+1}$.
- (b) Given $f(x) = \begin{cases} 5 - px & , \quad -2 < x \leq -1 \\ x^2 + px + q & , \quad -1 < x \leq 2 \\ \frac{x^2-4}{x-2} & , \quad x > 2 \end{cases}$
- (i) Find the values of p and q if $f(x)$ is continuous for all real values of x .
- (ii) Sketch the graph of $f(x)$ using the values p and q obtained in part (i).

SOLUTION

(a)

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2+3x}}{5x+1} &= \lim_{x \rightarrow -\infty} \frac{\sqrt{\frac{2x^2}{x^2} + \frac{3x}{x^2}}}{\frac{5x}{x} + \frac{1}{x}} \\ &= \lim_{x \rightarrow -\infty} \frac{\sqrt{2 + \frac{3}{x}}}{5 + \frac{1}{x}} \\ &= \frac{-\sqrt{2+0}}{5+0} \\ &= -\frac{\sqrt{2}}{5} \end{aligned}$$

(b)

$$f(x) = \begin{cases} 5 - px & , \quad -2 < x \leq -1 \\ x^2 + px + q & , \quad -1 < x \leq 2 \\ \frac{x^2-4}{x-2} & , \quad x > 2 \end{cases}$$

- (i) $f(x)$ is continuous at $x = -1$.

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x)$$

$$\lim_{x \rightarrow -1^-} 5 - px = \lim_{x \rightarrow -1^+} x^2 + px + q$$

$$5 - p(-1) = (-1)^2 + p(-1) + q$$

$$5 + p = 1 - p + q$$

$$2p - q = -4 \quad \dots\dots\dots (1)$$

$f(x)$ is continuous at $x = 2$.

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

$$\lim_{x \rightarrow 2^-} x^2 + px + q = \lim_{x \rightarrow 2^+} \frac{x^2 - 4}{x - 2}$$

$$2^2 + 2p + q = \lim_{x \rightarrow 2^+} \frac{(x + 2)(x - 2)}{x - 2}$$

$$4 + 2p + q = \lim_{x \rightarrow 2^+} (x + 2)$$

$$4 + 2p + q = 2 + 2$$

$$2p + q = 0 \quad \dots\dots\dots (2)$$

$$(2) - (1)$$

$$2q = 4$$

$$q = 2$$

$$p = -1$$

$$f(x) = \begin{cases} 5 + x & , -2 < x \leq -1 \\ x^2 - x + 2 & , -1 < x \leq 2 \\ \frac{x^2 - 4}{x - 2} & , x > 2 \end{cases}$$

$$= \begin{cases} 5 + x & , -2 < x \leq -1 \\ x^2 - x + 2 & , -1 < x \leq 2 \\ \frac{(x+2)(x-2)}{x-2} & , x > 2 \end{cases}$$

$$= \begin{cases} 5 + x & , -2 < x \leq -1 \\ x^2 - x + 2 & , -1 < x \leq 2 \\ x + 2 & , x > 2 \end{cases}$$

Sketch $y = x^2 - x + 2$

1. $a = 1, b = -1, c = 2$
2. $a > 0$ (open upwards)
3. Minimum point

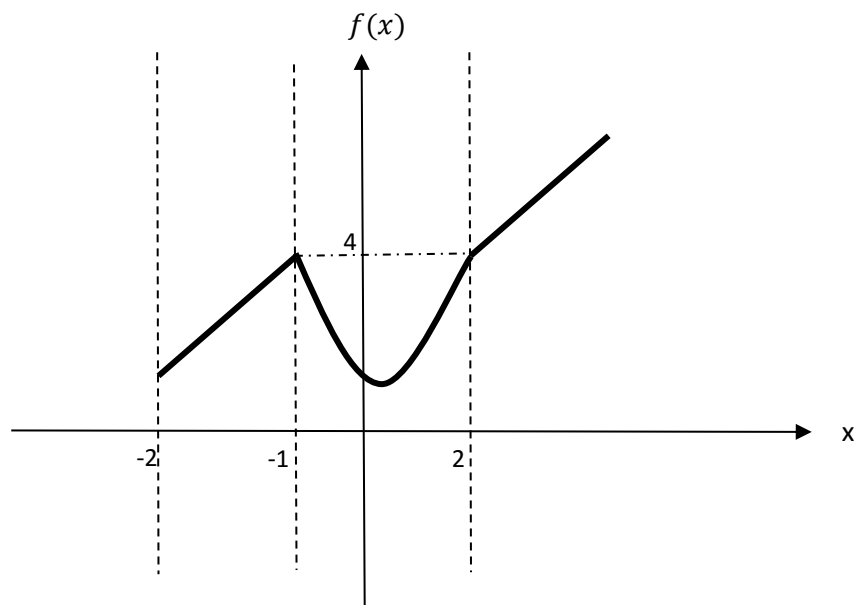
$$x = \frac{-b}{2a} = \frac{1}{2}$$

$$y = \frac{b^2 - 4ac}{-4a} = \frac{1 - 8}{-4} = \frac{7}{4}$$

4. Intercept

When $x = 0, y = 2$

(ii) Sketch the graph of $f(x)$



7. A curve is given by the parametric equations $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$.
- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .
 - Obtain the coordinates of the stationary points of the curve and determine the nature of the points.

SOLUTION**(a)**

$$x = t - \frac{1}{t} \qquad y = t + \frac{1}{t}$$

$$x = t - t^{-1} \qquad y = t + t^{-1}$$

$$\frac{dx}{dt} = 1 + t^{-2} \qquad \frac{dy}{dt} = 1 - t^{-2}$$

$$\frac{dx}{dt} = 1 + \frac{1}{t^2} \qquad \frac{dy}{dt} = 1 - \frac{1}{t^2}$$

$$\frac{dx}{dt} = \frac{t^2+1}{t^2} \qquad \frac{dy}{dt} = \frac{t^2-1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{t^2-1}{t^2} \cdot \frac{t^2}{t^2+1}$$

$$= \frac{t^2-1}{t^2+1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$$

$$\frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{t^2-1}{t^2+1} \right)$$

$$u = t^2 - 1 \qquad v = t^2 + 1$$

$$u' = 2t \qquad v' = 2t$$

$$\begin{aligned}\frac{d}{dt} \left(\frac{dy}{dx} \right) &= \frac{vu' - uv'}{v^2} \\ &= \frac{(t^2 + 1)(2t) - (t^2 - 1)(2t)}{(t^2 + 1)^2} \\ &= \frac{(2t^3 + 2t) - (2t^3 - 2t)}{(t^2 + 1)^2} \\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2} \\ &= \frac{4t}{(t^2 + 1)^2}\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\ &= \frac{4t}{(t^2 + 1)^2} \cdot \frac{t^2}{t^2 + 1} \\ &= \frac{4t^3}{(t^2 + 1)^3}\end{aligned}$$

(b)

To obtain the coordinates of the stationary points,

$$\text{Let } \frac{dy}{dx} = 0$$

$$\frac{t^2 - 1}{t^2 + 1} = 0$$

$$t^2 - 1 = 0$$

$$t^2 = 1$$

$$t = \pm 1$$

When $t = 1$

$$x = t - \frac{1}{t} \qquad y = t + \frac{1}{t}$$

$$x = 1 - \frac{1}{1} \qquad y = 1 + \frac{1}{1}$$

$$x = 0 \qquad y = 2$$

When $t = -1$

$$x = t - \frac{1}{t} \qquad y = t + \frac{1}{t}$$

$$x = -1 - \frac{1}{-1} \qquad y = -1 + \frac{1}{-1}$$

$$x = 0 \qquad y = -2$$

The stationary points are $(0, -2)$ and $(0, 2)$

When $t = 1$, and at the point $(0, 2)$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{4t^3}{(t^2 + 1)^3} \\ &= \frac{4(1)^3}{((1)^2 + 1)^3} \end{aligned}$$

$$> 0 \text{ (Min)}$$

When $t = -1$, and at the point $(0, -2)$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{4t^3}{(t^2 + 1)^3} \\ &= \frac{4(-1)^3}{((-1)^2 + 1)^3} \end{aligned}$$

$$< 0 \text{ (Max)}$$

$\therefore (0, -2)$ is a maximum point, $(0, 2)$ is a minimum point

8. (a) If $y^2 - 2y\sqrt{(1+x^2)} + x^2 = 0$, show that $\frac{dy}{dx} = \frac{x}{\sqrt{(1+x^2)}}$.
- (b) Water is running at a steady rate of $36\pi \text{ cm}^3\text{s}^{-1}$ into a right inverted circular cone with a semi-vertical angle of 45° .
- (i) Find the rate of increasing in water depth when the water level is 3 cm.
- (ii) Find the time taken when the depth of the water is 18cm.

SOLUTION

$$(a) \quad y^2 - 2y\sqrt{(1+x^2)} + x^2 = 0$$

$$y^2 - 2y(1+x^2)^{\frac{1}{2}} + x^2 = 0$$

$$2y \frac{dy}{dx} - \left[(2y) \left(\frac{1}{2} \right) (1+x^2)^{-\frac{1}{2}} (2x) + (1+x^2)^{\frac{1}{2}} \left(2 \frac{dy}{dx} \right) \right] + 2x = 0$$

$$2y \frac{dy}{dx} - \left[\frac{2xy}{\sqrt{1+x^2}} + 2\sqrt{1+x^2} \frac{dy}{dx} \right] + 2x = 0$$

$$2y \frac{dy}{dx} - \frac{2xy}{\sqrt{1+x^2}} - 2\sqrt{1+x^2} \frac{dy}{dx} + 2x = 0$$

$$2y \frac{dy}{dx} - 2\sqrt{1+x^2} \frac{dy}{dx} = \frac{2xy}{\sqrt{1+x^2}} - 2x$$

$$\frac{dy}{dx} (2y - 2\sqrt{1+x^2}) = \frac{2xy - 2x\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{2xy - 2x\sqrt{1+x^2}}{(2y - 2\sqrt{1+x^2})\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{2x(y - \sqrt{1+x^2})}{2(y - \sqrt{1+x^2})\sqrt{1+x^2}}$$

$$= \frac{x}{\sqrt{1+x^2}}$$

$$(b) \quad \frac{dv}{dt} = 36\pi \text{ cm}^3 \text{ s}^{-1}$$

(i) Find $\frac{dh}{dt}$ when $h = 3 \text{ cm}$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

$$v = \frac{1}{3}\pi r^2 h$$

Since $r = h$

$$v = \frac{1}{3}\pi h^2 h$$

$$= \frac{1}{3}\pi h^3$$

$$\frac{dv}{dh} = \pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$$

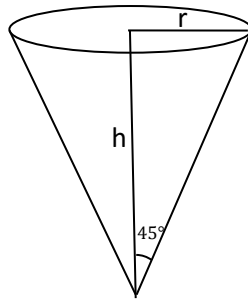
$$= \frac{1}{(\pi h^2)} \cdot (36\pi)$$

$$= \frac{36}{h^2}$$

When $h = 3 \text{ cm}$

$$\frac{dh}{dt} = \frac{36}{3^2}$$

$$= 4 \text{ cms}^{-1}$$



$$\tan 45^\circ = \frac{r}{h}$$

$$1 = \frac{r}{h}$$

$$h = r$$

(ii) Find t when $h = 18$ cm.

$$v = \frac{1}{3}\pi h^3$$

$$= \frac{1}{3}\pi(18)^3$$

$$= 1944\pi$$

$$\frac{dv}{dt} = 36\pi$$

$$\frac{1944\pi}{t} = 36\pi$$

$$t = 54\text{s}$$

9. (a) Determine the values of R and α , where $R > 0$ and $0^\circ < \alpha < 90^\circ$ so that

$$3 \sin \theta - 4 \cos \theta = R \sin(\theta - \alpha).$$
- (b) Hence, solve the equation $3 \sin \theta - 4 \cos \theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$.
- (c) From the answer obtained in part (b), find the value of θ for $0^\circ \leq \theta \leq 360^\circ$ so that

$$f(\theta) = \frac{1}{3 \sin \theta - 4 \cos \theta + 15}$$
 is minimum. Hence, find the minimum value of f .

SOLUTION**(a)**

$$3 \sin \theta - 4 \cos \theta = R \sin(\theta - \alpha).$$

$$3 \sin \theta - 4 \cos \theta = R(\sin \theta \cos \alpha - \cos \theta \sin \alpha)$$

$$3 \sin \theta - 4 \cos \theta = R \sin \theta \cos \alpha - R \cos \theta \sin \alpha$$

$$\sin \theta: \quad R \cos \alpha = 3 \quad \dots\dots\dots (1)$$

$$\cos \theta: \quad R \sin \alpha = 4 \quad \dots\dots\dots (2)$$

$$(1)^2 + (2)^2$$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 3^2 + 4^2$$

$$R^2(\cos^2 \alpha + \sin^2 \alpha) = 25$$

$$R^2 = 25$$

$$R = 5$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$(2) \div (1)$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{4}{3}$$

$$\frac{\sin \alpha}{\cos \alpha} = \frac{4}{3}$$

$$\tan \alpha = \frac{3}{4}$$

$$\alpha = 53.1^\circ$$

$$\therefore 3 \sin \theta - 4 \cos \theta = 5 \sin(\theta - 53.1^\circ)$$

(b)

$$3 \sin \theta - 4 \cos \theta = 2$$

$$0^\circ \leq \theta \leq 360^\circ$$

$$5 \sin(\theta - 53.1^\circ) = 2$$

$$\sin(\theta - 53.1^\circ) = \frac{2}{5}$$

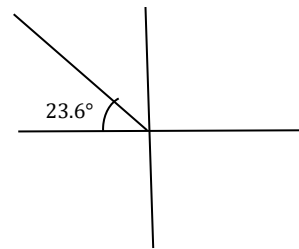
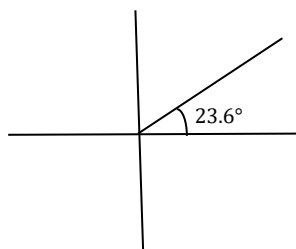
$$\sin(\theta - 53.1^\circ) = 0.4$$

$$0^\circ \leq \theta \leq 360^\circ$$

$$0^\circ - 53.1^\circ \leq \theta - 53.1^\circ \leq 360^\circ - 53.1^\circ$$

$$-53.1^\circ \leq \theta - 53.1^\circ \leq 306.9^\circ$$

$$\sin(\theta - 53.1^\circ) = 0.4$$



$$\theta - 53.1^\circ = 23.6^\circ, 180^\circ - 23.6^\circ$$

$$\theta - 53.1^\circ = 23.6^\circ, 156.4^\circ$$

$$\theta = 23.6^\circ + 53.1^\circ, 156.4^\circ + 53.1^\circ$$

$$\theta = 76.7^\circ, 209.5^\circ$$

(c)

$$f(\theta) = \frac{1}{3\sin\theta - 4\cos\theta + 15}$$

$$f(\theta) = \frac{1}{5\sin(\theta - 53.1^\circ) + 15}$$

Since $-1 \leq \sin(\theta - 53.1^\circ) \leq 1$

For $f(\theta)$ minimum

$$\sin(\theta - 53.1^\circ) = 1$$

$$\theta - 53.1^\circ = 90^\circ$$

$$\theta = 143.1^\circ$$

Therefore, the minimum value of $f(\theta)$

$$\begin{aligned} f(\theta) &= \frac{1}{5\sin(143.1^\circ - 53.1^\circ) + 15} \\ &= \frac{1}{5\sin(90^\circ) + 15} \\ &= \frac{1}{5(1) + 15} \\ &= \frac{1}{20} \end{aligned}$$

10. (a) Find the value of k if the slope of the curve $x^3 + kx^2y - 2y^2 = 0$ at the point $(-1,1)$ is -3 .
- (b) Given $y = \frac{\sin x}{1+\cos x}$.
- (i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x .
- (ii) Hence, show that $\frac{d^3y}{dx^3} - y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$.

SOLUTION**(a)**

$$x^3 + kx^2y - 2y^2 = 0$$

$$3x^2 + \left[kx^2 \frac{dy}{dx} + 2kxy \right] - 4y \frac{dy}{dx} = 0$$

$$3x^2 + kx^2 \frac{dy}{dx} + 2kxy - 4y \frac{dy}{dx} = 0$$

$$kx^2 \frac{dy}{dx} - 4y \frac{dy}{dx} = -2kxy - 3x^2$$

$$\frac{dy}{dx} [kx^2 - 4y] = -2kxy - 3x^2$$

$$\frac{dy}{dx} = \frac{-2kxy - 3x^2}{kx^2 - 4y}$$

At the point $(-1,1) \rightarrow x = -1, y = 1 \quad \frac{dy}{dx} = -3$

$$\frac{dy}{dx} = \frac{-2kxy - 3x^2}{kx^2 - 4y}$$

$$-3 = \frac{-2k(-1)(1) - 3(-1)^2}{k(-1)^2 - 4(1)}$$

$$-3 = \frac{2k - 3}{k - 4}$$

$$-3(k - 4) = 2k - 3$$

$$-3k + 12 = 2k - 3$$

$$12 + 3 = 2k + 3k$$

$$5k = 15$$

$$k = 3$$

(b)

(i) $y = \frac{\sin x}{1 + \cos x}$

$$u = \sin x \qquad v = 1 + \cos x$$

$$u' = \cos x \qquad v' = -\sin x$$

$$\frac{dy}{dx} = \frac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2}$$

$$= \frac{1}{1 + \cos x}$$

$$= (1 + \cos x)^{-1}$$

$$\frac{d^2y}{dx^2} = -(1 + \cos x)^{-2} \frac{d}{dx} (1 + \cos x)$$

$$= -(1 + \cos x)^{-2} (-\sin x)$$

$$= \frac{\sin x}{(1 + \cos x)^2}$$

(ii) show that $\frac{d^3y}{dx^3} - y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$

$$\frac{d^2y}{dx^2} = \frac{\sin x}{(1 + \cos x)^2}$$

$$u = \sin x \qquad v = (1 + \cos x)^2$$

$$\begin{aligned} u' &= \cos x & v' &= 2(1 + \cos x) \frac{d}{dx}(1 + \cos x) \\ & & &= 2(1 + \cos x)(-\sin x) \\ & & &= -2 \sin x (1 + \cos x) \end{aligned}$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{(1 + \cos x)^2(\cos x) - [-2 \sin x (1 + \cos x)](\sin x)}{[(1 + \cos x)^2]^2} \\ &= \frac{(1 + \cos x)^2(\cos x) + 2 \sin^2 x (1 + \cos x)}{(1 + \cos x)^4} \\ &= \frac{(1 + \cos x)[\cos x (1 + \cos x) + 2 \sin^2 x]}{(1 + \cos x)^4} \\ &= \frac{\cos x + \cos^2 x + 2 \sin^2 x}{(1 + \cos x)^3} \end{aligned}$$

$$\begin{aligned} \frac{d^3y}{dx^3} - y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 &= \left(\frac{\cos x + \cos^2 x + 2 \sin^2 x}{(1 + \cos x)^3}\right) - y \left(\frac{\sin x}{(1 + \cos x)^2}\right) - \left(\frac{1}{1 + \cos x}\right)^2 \\ &= \frac{\cos x + \cos^2 x + 2 \sin^2 x}{(1 + \cos x)^3} - \frac{y \sin x}{(1 + \cos x)^2} - \frac{1}{(1 + \cos x)^2} \\ &= \frac{\cos x + \cos^2 x + 2 \sin^2 x}{(1 + \cos x)^3} - \frac{\left(\frac{\sin x}{1 + \cos x}\right) \sin x}{(1 + \cos x)^2} - \frac{1}{(1 + \cos x)^2} \\ &= \frac{\cos x + \cos^2 x + 2 \sin^2 x}{(1 + \cos x)^3} - \frac{\sin^2 x}{(1 + \cos x)^3} - \frac{1 + \cos x}{(1 + \cos x)^3} \end{aligned}$$

$$= \frac{\cos x + \cos^2 x + 2 \sin^2 x - \sin^2 x - (1 + \cos x)}{(1 + \cos x)^3}$$

$$= \frac{\cos x + \cos^2 x + 2 \sin^2 x - \sin^2 x - 1 - \cos x}{(1 + \cos x)^3}$$

$$= \frac{\cos^2 x + \sin^2 x - 1}{(1 + \cos x)^3}$$

$$= \frac{1 - 1}{(1 + \cos x)^3}$$

$$= 0$$

