KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025

Mid-Semester Examination

Semester II

Session 2011/2012

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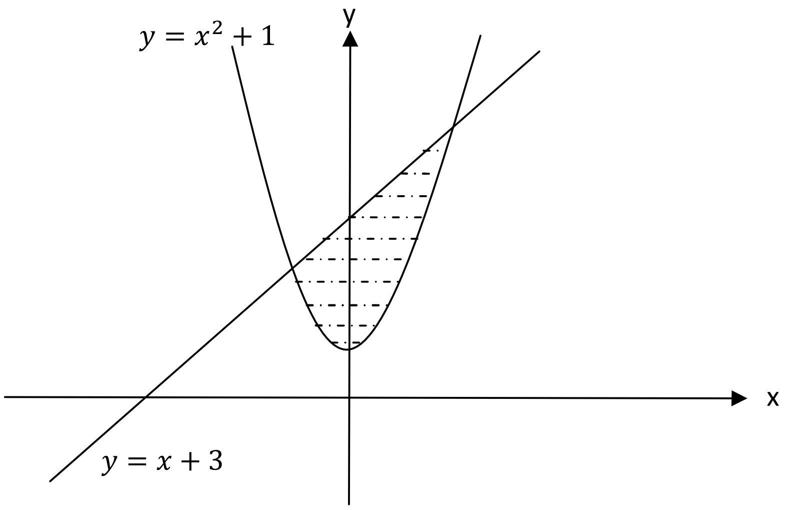
1. Find the area enclosed by the graph $y = x + 3$ and $y = x^2 + 1$.
2. Find
   a. $\int (3 + e^x)(2 + e^{-x})\, dx$
   b. $\int x \cos(2 - 4x^2)\, dx$ by using an appropriate substitution method.
3. Given the equation $e^x = \frac{2}{x} + 1$.
   a. Show that there is a real root between 1 and 2.
   b. By using Newton-Raphson method, find the root of the equation correct to three decimal places, taking 1.5 as the first approximation.
4. Solve the differential equation $\frac{dy}{dx} = -2y + 3$ by using integrating factor method. Express y in terms of x.
5. Express $\frac{3x^2-7x+6}{(x-3)^2(x+1)}$ in the form of partial fraction. Hence, evaluate $\int_1^2 \frac{3x^2-7x+6}{(x-3)^2(x+1)}\, dx$. Give the answer in the form of $a + \ln b$.
6. The Cartesian equation of an ellipse is given by $4x^2 + 9y^2 - 16x + 36y + 16 = 0$.
   a. Express the Cartesian equation of the ellipse in the standard form.
   b. Determine the coordinates of the centre, vertices and foci of the ellipse.
   c. Sketch the graph of the ellipse.

1. Find the area enclosed by the graph $y = x + 3$ and $y = x^2 + 1$.

SOLUTION



Intersection points:

$x^2 + 1 = x + 3$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1$ or $x = 2$

$$Area = \int_{-1}^{2} (x + 3) - (x^2 + 1)\, dx$$

$$= \int_{-1}^{2} -x^2 + x + 2\ dx$$

$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2}$$

$$= \left[-\frac{(2)^3}{3} + \frac{(2)^2}{2} + 2(2)\right] - \left[-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1)\right]$$

$$= \left[-\frac{8}{3} + \frac{4}{2} + 4\right] - \left[\frac{1}{3} + \frac{1}{2} - 2\right]$$

$$= \frac{9}{2}\ unit^2$$

2. Find
   a. $\int (3+e^x)(2+e^{-x})\,dx$
   b. $\int x\cos(2-4x^2)\,dx$ by using an appropriate substitution method.

SOLUTION

(a)

$$\int (3+e^x)(2+e^{-x})\,dx = \int 6+3e^{-x}+2e^x+e^xe^{-x}dx$$

$$= \int 6+3e^{-x}+2e^x+e^0dx$$

$$= \int 7+3e^{-x}+2e^xdx$$

$$= 7x-3e^{-x}+2e^x+c$$

(b)

$$\int x\cos(2-4x^2)\,dx$$

$$u = 2-4x^2$$

$$\frac{du}{dx} = -8x$$

$$dx = -\frac{1}{8x}du$$

$$\int x\cos(2-4x^2)\,dx = \int x\ \cos u\left(-\frac{1}{8x}du\right)$$

$$= -\frac{1}{8}\int \cos u\,du$$

$$= -\frac{1}{8}\sin u + C$$

$$= -\frac{1}{8}\sin(2-4x^2) + C$$

3. Given the equation $e^x = \frac{2}{x} + 1$.
   a. Show that there is a real root between 1 and 2.
   b. By using Newton-Raphson method, find the root of the equation correct to three decimal places, taking 1.5 as the first approximation.

SOLUTION

(a)

$$e^x = \frac{2}{x} + 1$$

$$f(x) = e^x - \frac{2}{x} - 1$$

$$f(1) = e^1 - \frac{2}{1} - 1 = -0.2817 < 0$$

$$f(2) = e^2 - \frac{2}{2} - 1 = 5.39 > 0$$

Since $f(1) < 0$ and $f(2) > 0$, therefore root lies between 1 and 2.

(b)

$$f(x) = e^x - \frac{2}{x} - 1$$

$$f'(x) = e^x + \frac{2}{x^2}$$

$$x_{n+1} = x_n - \frac{e^{x_n} - \frac{2}{x_n} - 1}{e^{x_n} + \frac{2}{{x_n}^2}}$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

$$x_0 = 1.5$$

$$x_1 = 1.5 - \frac{e^{1.5} - \frac{2}{1.5} - 1}{e^{1.5} + \frac{2}{(1.5)^2}} = 1.0999$$

$$x_2 = 1.0999 - \frac{e^{1.0999} - \frac{2}{1.0999} - 1}{e^{1.0999} + \frac{2}{(1.0999)^2}} = 1.0600$$

$$x_3 = 1.0600 - \frac{e^{1.0600} - \frac{2}{1.0600} - 1}{e^{1.0600} + \frac{2}{(1.0600)^2}} = 1.0601$$

$$x_4 = 1.0601 - \frac{e^{1.0601} - \frac{2}{1.0601} - 1}{e^{1.0601} + \frac{2}{(1.0601)^2}} = 1.0601$$

$$\therefore \textit{The root is } 1.060$$

4. Solve the differential equation $\frac{dy}{dx} = -2y + 3$ by using integrating factor method. Express y in terms of x.

SOLUTION

$$\frac{dy}{dx} = -2y + 3$$

$$\frac{dy}{dx} + 2y = 3$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Integrating factor,

$$P(x) = 2; \quad Q(x) = 3$$

$$V(x) = e^{\int P(x)dx}$$

$$V(x) = e^{\int 2dx} = e^{2x}$$

$$V(x)y = \int V(x)Q(x)dx$$

$$e^{2x}y = \int e^{2x}(3)dx$$

$$ye^{2x} = 3\int e^{2x}\,dx$$

$$ye^{2x} = 3\left(\frac{e^{2x}}{2}\right) + c$$

$$ye^{2x} = \frac{3e^{2x}}{2} + c$$

$$y = \frac{3e^{2x}}{2(e^{2x})} + \frac{c}{e^{2x}}$$

$$y = \frac{3}{2} + ce^{-2x}$$

5. Express $\frac{3x^2-7x+6}{(x-3)^2(x+1)}$ in the form of partial fraction. Hence, evaluate $\int_1^2 \frac{3x^2-7x+6}{(x-3)^2(x+1)}\,dx$. Give the answer in the form of $a + \ln b$.

SOLUTION

$$\frac{3x^2-7x+6}{(x-3)^2(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x+1)}$$

$$\frac{3x^2-7x+6}{(x-3)^2(x+1)} = \frac{A(x-3)(x+1) + B(x+1) + C(x-3)^2}{(x-3)^2(x+1)}$$

$$3x^2 - 7x + 6 = A(x-3)(x+1) + B(x+1) + C(x-3)^2$$

When x = 3:

$$3(3)^2 - 7(3) + 6 = B(3+1)$$

$$27 - 21 + 6 = 4B$$

$$4B = 12$$

$$B = 3$$

When x = -1:

$$3(-1)^2 - 7(-1) + 6 = C[(-1)-3]^2$$

$$16 = 16C$$

$$C = 1$$

When x = 0, B=3, C=1:

$$6 = A(-3)(1) + 3(1) + 1(-3)^2$$

$$6 = -3A + 3 + 9$$

$$3A = 6$$

$$A = 2$$

$$\frac{3x^2-7x+6}{(x-3)^2(x+1)} = \frac{2}{(x-3)} + \frac{3}{(x-3)^2} + \frac{1}{(x+1)}$$

$$\int \frac{3x^2-7x+6}{(x-3)^2(x+1)}\,dx = \int \frac{2}{(x-3)} + \frac{3}{(x-3)^2} + \frac{1}{(x+1)}\,dx$$

$$= \int \frac{2}{(x-3)}\,dx + \int \frac{3}{(x-3)^2}\,dx + \int \frac{1}{(x+1)}\,dx$$

$$= 2\int \frac{1}{(x-3)}dx + 3\int (x-3)^{-2}dx + \int \frac{1}{(x+1)}dx$$

$$= 2\ln(x-3) + (3)\ \frac{(x-3)^{-2+1}}{-1} + \ln(x+1)$$

$$= 2\ln(x-3) - \frac{3}{(x-3)} + \ln(x+1)$$

$$= \ln(x-3)^2 - \frac{3}{(x-3)} + \ln(x+1)$$

$$= \ln[(x-3)^2(x+1)] - \frac{3}{(x-3)}$$

$$\int_1^2 \frac{3x^2-7x+6}{(x-3)^2(x+1)}\,dx = \left[\ln[(x-3)^2(x+1)] - \frac{3}{(x-3)}\right]_1^2$$

$$= \left[\ln[(2-3)^2(2+1)] - \frac{3}{(2-3)}\right] - \left[\ln[(1-3)^2(1+1)] - \frac{3}{(1-3)}\right]$$

$$= \left[\ln 3 - \frac{3}{(2-3)}\right] - \left[\ln 8 - \frac{3}{(1-3)}\right]$$

$$= \ln 3 - \ln 8 - \frac{3}{2}$$

$$= \ln\left(\frac{3}{8}\right) + \frac{3}{2}$$

$$= \frac{3}{2} + \ln\left(\frac{3}{8}\right)$$

6. The Cartesian equation of an ellipse is given by $4x^2 + 9y^2 - 16x + 36y + 16 = 0$.
   a. Express the Cartesian equation of the ellipse in the standard form.
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SOLUTION

(a)

$$4x^2 + 9y^2 - 16x + 36y + 16 = 0$$

$$[4x^2 - 16x] + [9y^2 + 36y] + 16 = 0$$

$$4[x^2 - 4x] + 9[y^2 + 4y] + 16 = 0$$

$$4\left[x^2 - 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] + 9\left[y^2 + 4y + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2\right] + 16 = 0$$

$$4[x^2 - 4x + 2^2 - 2^2] + 9[y^2 + 4y + 2^2 - 2^2] + 16 = 0$$

$$4[(x-2)^2 - 4] + 9[(y+2)^2 - 4] + 16 = 0$$

$$4(x-2)^2 - 16 + 9(y+2)^2 - 36 + 16 = 0$$

$$4(x-2)^2 + 9(y+2)^2 = 36$$

$$\frac{4(x-2)^2}{36} + \frac{9(y+2)^2}{36} = \frac{36}{36}$$

$$\frac{(x-2)^2}{9} + \frac{(y+2)^2}{4} = 1$$

(b)

$$\frac{(x-2)^2}{9} + \frac{(y+2)^2}{4} = 1$$

Compare to

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$Centre(h, k) = (2, -2)$$

$$c^2 = a^2 - b^2$$

$$c^2 = 9 - 4 = 5$$

$$c = \sqrt{5}$$

$$Vertices = (2 \pm 3, -2) = (5, -2); (-1, -2)$$

$$Foci = (2 \pm \sqrt{5}, -2) = (2 + \sqrt{5}, -2); (2 - \sqrt{5}, -2)$$

(c)