KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025

Mid-Semester Examination

Semester II

Session 2012/2013

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1) Find

a) $\int_0^1 \frac{3+e^x}{e^x}\,dx$.

b) $\int \sin^2 x\ \cos x\ dx$

2) Using integration by parts, solve

$$\int 2x\ \sin 2x\ dx$$

3) Find the particular solution of the differential equation

$$(x+2)\frac{dy}{dx} = y+2, \qquad y(0) = 1$$

4) Show that the equation $2x + e^x - 2 = 0$ has a root between x = 0 and x = 1. Using the Newton-Raphson method and taking $x_0 = 0.7$, find the root correct to four decimal places.

5) (a) Using partial fractions, find $\int \frac{3x^2-2x+2}{(x-1)(x^2+2)}\,dx$.

(b) Given functions $f(x) = x + 7$ and $g(x) = x^2 - 5$, find the area of the region bounded by $f(x) and\ g(x)$.

6) A circle passes through the points $A(-2, 4)$, $B(7, 7)$ and the centre lies on the line $y - x = 1$. Find the radius and standard equation of the circle.

END OF QUESTION PAPER

1) Find

a) $\int_0^1 \frac{3+e^x}{e^x} dx.$

b) $\int \sin^2 x \ \cos x \ dx$

SOLUTION

(a)

$$\int \frac{3+e^x}{e^x} dx = \int \frac{3}{e^x} + \frac{e^x}{e^x} dx$$

$$= \int (3e^{-x} + 1) dx$$

$$= -3e^{-x} + x$$

$$= \frac{-3}{e^x} + x$$

$$\int_0^1 \frac{3+e^x}{e^x} dx = \left[\frac{-3}{e^x} + x\right]_0^1$$

$$= \left[\frac{-3}{e^1} + 1\right] - \left[\frac{-3}{e^0} + 0\right]$$

$$= \left[\frac{-3}{e} + 1\right] - \left[\frac{-3}{1} + 0\right]$$

$$= \frac{-3}{e} + 1 + 3$$

$$= 4 - \frac{3}{e}$$

(b)

$$\int \sin^2 x \ \cos x \ dx$$

$$u = sinx$$

$$\frac{du}{dx} = \cos x$$

$$dx = \frac{1}{\cos x} du$$

$$\int \sin^2 x \ \cos x \ dx = \int u^2 \ \cos x \left(\frac{1}{\cos x} du\right)$$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{sin^3 x}{3} + c$$

2) Using integration by parts, solve

$$\int 2x \ \sin 2x \ dx$$

SOLUTION

$$\int 2x \ \sin 2x \ dx$$

$u = 2x$ $\qquad\qquad$ $dv = \sin 2x \, dx$

$\frac{du}{dx} = 2$ $\qquad\qquad$ $\int dv = \int \sin 2x \, dx$

$du = 2dx$ $\qquad\qquad$ $v = \frac{-\cos 2x}{2}$

$$\int udv = uv - \int vdu$$

$$\int 2x \ \sin 2x \ dx = (2x)\left(\frac{-\cos 2x}{2}\right) - \int \left(\frac{-\cos 2x}{2}\right)(2dx)$$

$$= -\text{x}\cos 2x + \int \cos 2x \, dx$$

$$= -\text{x}\cos 2x + \frac{\sin 2x}{2} + c$$

3) Find the particular solution of the differential equation

$(x+2)\frac{dy}{dx} = y+2, \qquad y(0) = 1$

SOLUTION

$$(x+2)\frac{dy}{dx} = y+2$$

$$(x+2)dy = (y+2)dx$$

$$\frac{1}{(y+2)}dy = \frac{1}{(x+2)}dx$$

$$\int \frac{1}{(y+2)}dy = \int \frac{1}{(x+2)}dx$$

$$\ln|y+2| = \ln|x+2| + c$$

Given that $y(0) = 1$ ➔ when x = 0, y = 1

$$\ln|1+2| = \ln|0+2| + c$$

$$\ln|3| = \ln|2| + c$$

$$c = \ln|3| - \ln|2|$$

$$c = \ln\left(\frac{3}{2}\right)$$

$\log m - lon\, n = \log\left(\frac{m}{n}\right)$

$$\ln|y+2| = \ln|x+2| + c$$

$$\ln|y+2| = \ln|x+2| + \ln\left(\frac{3}{2}\right)$$

$$\ln|y+2| = \ln\frac{3|x+2|}{2}$$

$\log m + log\, n = \log(mn)$

$$y+2 = \frac{3}{2}(x+2)$$

$$y = \frac{3}{2}(x+2) - 2$$

$$y = \frac{3x}{2} + 3 - 2$$

$$y = \frac{3x}{2} + 1$$

4) Show that the equation $2x + e^x - 2 = 0$ has a root between x = 0 and x = 1. Using the Newton-Raphson method and taking $x_0 = 0.7$, find the root correct to four decimal places.

SOLUTION

$2x + e^x - 2 = 0$

$f(x) = 2x + e^x - 2$

$f(0) = 2(0) + e^{(0)} - 2 = -1 < 0$

$f(1) = 2(1) + e^{(1)} - 2 = e > 0$

Since $f(0) < 0$ and $f(1) > 0$. $\therefore f(x)$ has a root between $x = 0 \; and \; x = 1$.

$f(x) = 2x + e^x - 2$

$f'(x) = 2 + e^x$

$$x_0 = 0.7$$

$$x_1 = 0.7 - \frac{2(0.7) + e^{0.7} - 2}{2 + e^{0.7}} = 0.34777$$

$$x_2 = 0.34777 - \frac{2(0.34777) + e^{0.34777} - 2}{2 + e^{0.34777}} = 0.31514$$

$$x_3 = 0.31514 - \frac{2(0.31514) + e^{0.31514} - 2}{2 + e^{0.31514}} = 0.31492$$

$$x_4 = 0.31492 - \frac{2(0.31492) + e^{0.31492} - 2}{2 + e^{0.31492}} = 0.31492$$

$\therefore \textit{The root is } 0.3149$

5) (a) Using partial fractions, find $\int \frac{3x^2-2x+2}{(x-1)(x^2+2)} dx$.

(b) Given functions $f(x) = x + 7$ and $g(x) = x^2 - 5$, find the area of the region bounded by $f(x) and\ g(x)$.

SOLUTION

(a)

$$\frac{3x^2-2x+2}{(x-1)(x^2+2)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+2)}$$

$$\frac{3x^2-2x+2}{(x-1)(x^2+2)} = \frac{A(x^2+2)+(Bx+C)(x-1)}{(x-1)(x^2+2)}$$

$$3x^2-2x+2 = A(x^2+2)+(Bx+C)(x-1)$$

When x = 1

$$3(1)^2-2(1)+2 = A[(1)^2+2]+[B(1)+C][(1)-1]$$

$$3 = 3A$$

$$A = 1$$

When x = 0, A = 1

$$3(0)^2-2(0)+2 = 1[(0)^2+2]+[B(0)+C][(0)-1]$$

$$2 = 2 - C$$

$$C = 0$$

When x = -1, A = 1, C = 0

$$3(-1)^2-2(-1)+2 = 1[(-1)^2+2]+[B(-1)+0][(-1)-1]$$

$$7 = 3 + [-B][-2]$$

$$4 = 2B$$

$$B = 2$$

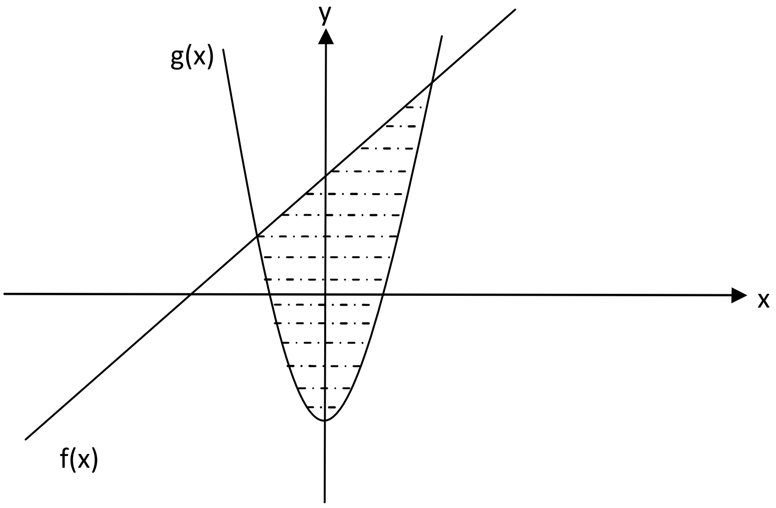
$$\frac{3x^2-2x+2}{(x-1)(x^2+2)} = \frac{1}{(x-1)} + \frac{2x}{(x^2+2)}$$

$$\int \frac{3x^2-2x+2}{(x-1)(x^2+2)} dx = \int \frac{1}{(x-1)} + \frac{2x}{(x^2+2)} dx$$

$$= \ln(x-1) + \ln(x^2+2) + C$$

$(\boldsymbol{b})$

$f(x) = x + 7$ and $g(x) = x^2 - 5$



$f(x) = x + 7$ .... (1)

$g(x) = x^2 - 5$ .... (2)

To find intersection points, let $f(x) = g(x)$.

$x + 7 = x^2 - 5$

$x^2 - x - 12 = 0$

$(x - 4)(x + 3) = 0$

$x = 4$ or $x = -3$

$$Area = \int_{-3}^{4} (x + 7) - (x^2 - 5)\ dx$$

$$= \int_{-3}^{4} -x^2 + x + 12\ dx$$

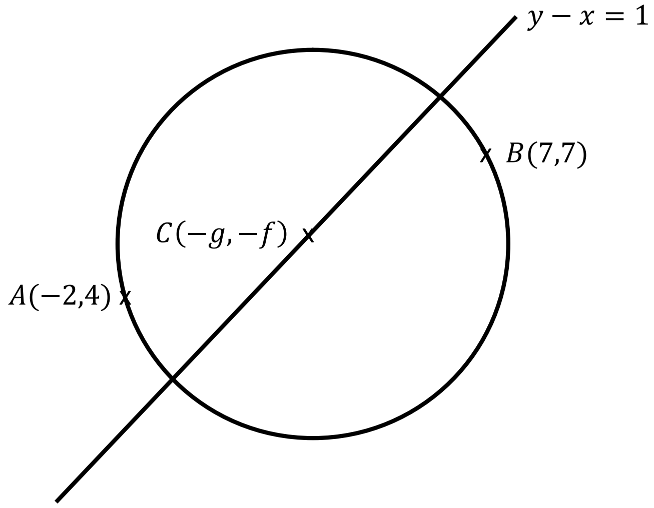
$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 12x\right]_{-3}^{4}$$

$$= \left[-\frac{4^3}{3} + \frac{4^2}{2} + 12(4)\right] - \left[-\frac{(-3)^3}{3} + \frac{(-3)^2}{2} + 12(-3)\right]$$

$$= \frac{343}{6}\, unit^2$$

6) A circle passes through the points $A(-2, 4)$, $B(7, 7)$ and the centre lies on the line $y - x = 1$. Find the radius and standard equation of the circle.

SOLUTION



General Equation:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

At the point $A(-2,4)$ ➔ $x = -2;\quad y = 4$

$$(-2)^2 + (4)^2 + 2g(-2) + 2f(4) + c = 0$$

$$4 + 16 - 4g + 8f + c = 0$$

$$4g - 8f - c = 20 \qquad \text{.................. (1)}$$

At the point $B(7, 7)$ ➔ $x = 7;\quad y = 7$

$$(7)^2 + (7)^2 + 2g(7) + 2f(7) + c = 0$$

$$49 + 49 + 14g + 14f + c = 0$$

$$14g + 14f + c = -98 \qquad \text{.................. (2)}$$

Center $C(-g, -f)$ lies on the line $y - x = 1$ ➔ $x = -g,\quad y = -f$

$$-f - (-g) = 1$$

$$g - f = 1 \qquad \text{.................. (3)}$$

(2) – (1)

$$10g + 22f = -118 \qquad \text{.................. (4)}$$

$(3)x\ 10$

$10g - 10f = 10$ .... (5)

$(4) - (5)$

$32f = -128$

$f = -4$

$10g - 10(-4) = 10$

$10g = -30$

$g = -3$

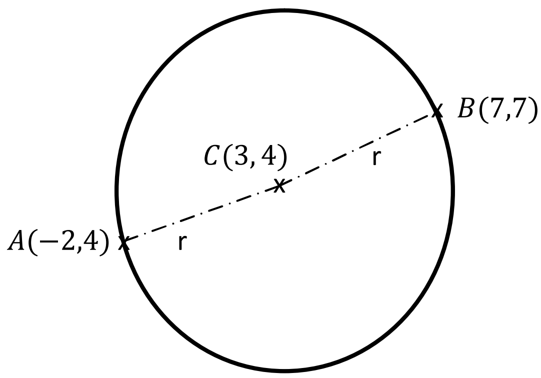
$4g - 8f - c = 20$

$4(-3) - 8(-4) - c = 20$

$20 - c = 20$

$c = 0$

$\therefore Center\ (3, 4)$



$A(-2, 4), Center\ (3, 4)$

$radius = \sqrt{(3+2)^2 + (4-4)^2}$

$= 5$

$Standard\ equation\ of\ circle:$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 3)^2 + (y - 4)^2 = 5^2$

$(x - 3)^2 + (y - 4)^2 = 25$