KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025

Mid-Semester Examination

Semester II

Session 2013/2014

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1. By substituting $u = 1 + e^x$, evaluate $\int_0^1 \frac{e^x}{1+e^x} dx$. Give your answer in terms of e.

2. Evaluate $\int_1^2 x^2 ln3xdx$ correct to three decimal places.

3. By using separable variable method, find the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{2(x-1)}$. Hence, determine the particular solution if y = 2 when x=5.

4. Given $e^x = 4 - x$.
   a) Show that, there is a real root between 1 and 2.
   b) Hence, by using the Newton-Raphson method, find the root of the equation correct to four decimal places. Given that $x_0 = 1.2$ as the first approximation.

5. (a) Find the area of the region bounded by the curve $x = y^2$ and the straight line $y + x - 2 = 0$.
   (b) The region bounded by $y = x^2 + 3x, x = -3$ and $x = -1$ is rotated completely about the x-axis. Find the volume of the solid formed.

6. (a) A circle with center (4, -2) passes through the points (10, 6) and (a, 8). Find
   i. The value of a.
   ii. The general equation of the circle.

   (b) Find the standard equation of a parabola with its symmetric axis parrallel to the x-axis, vertex at the point (3, 2) and passing through the point (4, 4).

1. By substituting $u = 1 + e^x$, evaluate $\int_0^1 \frac{e^x}{1+e^x} dx$. Give your answer in terms of e.

SOLUTION

$u = 1 + e^x$

$\frac{du}{dx} = e^x \rightarrow dx = \frac{1}{e^x}.du$

$$\int \frac{e^x}{1+e^x} dx = \int \frac{e^x}{u} . \frac{1}{e^x} . du$$

$$= \int \frac{1}{u} . du$$

$$= \ln u$$

$$= \ln (1 + e^x)$$

$$\int_0^1 \frac{e^x}{1+e^x} dx = [\ln (1 + e^x)]_0^1$$

$$= [\ln (1 + e^1)] - [\ln (1 + e^0)]$$

$$= [\ln (1 + e)] - [\ln (1 + 1)]$$

$$= [\ln (1 + e)] - [\ln (2)]$$

$$\boxed{\log M - \log N = \log \frac{M}{N}}$$

$$= \ln \left[\frac{1+e}{2}\right]$$

2. Evaluate $\int_1^2 x^2 ln3xdx$ correct to three decimal places.

SOLUTION

$$\int x^2 ln(3x)dx$$

$u = ln(3x)$ $\qquad$ $dv = x^2dx$

$\frac{du}{dx} = \frac{1}{3x}.\frac{d}{dx}(3x)$ $\qquad$ $\int dv = \int x^2dx$

$\frac{du}{dx} = \frac{1}{3x}.(3)$ $\qquad$ $v = \frac{x^3}{3}$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}\,dx$$

$\int x^2 ln(3x)dx = uv - \int vdu$

$= [ln(3x)]\left[\frac{x^3}{3}\right] - \int \frac{x^3}{3}.\frac{1}{x}\,dx$

$= \frac{x^3 ln(3x)}{3} - \frac{1}{3}\int x^2\,dx$

$= \frac{x^3 ln(3x)}{3} - \frac{1}{3}\left[\frac{x^3}{3}\right]$

$= \frac{x^3 ln(3x)}{3} - \frac{x^3}{9}$

$\int_1^2 x^2 ln3xdx = \left[\frac{x^3 ln(3x)}{3} - \frac{x^3}{9}\right]_1^2$

$= \left[\frac{2^3 ln(3x2)}{3} - \frac{2^3}{9}\right] - \left[\frac{1^3 ln(3x1)}{3} - \frac{1^3}{9}\right]$

$= \left[\frac{8ln(6)}{3} - \frac{8}{9}\right] - \left[\frac{ln(3)}{3} - \frac{1}{9}\right]$

$=3.634$

3. By using separable variable method, find the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{2(x-1)}$. Hence, determine the particular solution if y = 2 when x=5.

SOLUTION

$$\frac{dy}{dx} = \frac{y}{2(x-1)}$$

$$\frac{1}{y}dy = \frac{1}{2(x-1)}dx$$

$$\int \frac{1}{y}dy = \int \frac{1}{2(x-1)}dx$$

$$\ln y = \frac{1}{2}\ln(x-1) + c$$

General Solution

When y = 2, x = 5

$$\ln 2 = \frac{1}{2}\ln(5-1) + c$$

$$\ln 2 = \frac{1}{2}\ln\ 4 + c$$

$$c = \ln 2 - \frac{1}{2}\ln\ 4$$

$$c = \ln 2 - \ln\ 4^{\frac{1}{2}}$$

$$c = \ln 2 - \ln\ 2$$

$$c = 0$$

Particular Solution

$$\ln y = \frac{1}{2}\ln(x-1) + 0$$

$$\ln y = \frac{1}{2}\ln(x-1)$$

$m \log n = \log n^m$

$$\ln y = \ln(x-1)^{\frac{1}{2}}$$

$$y = (x-1)^{\frac{1}{2}}$$

4. Given $e^x = 4 - x$.

a) Show that, there is a real root between 1 and 2.

b) Hence, by using the Newton-Raphson method, find the root of the equation correct to four decimal places. Given that $x_0 = 1.2$ as the first approximation.

SOLUTION

(a)

$$e^x = 4 - x$$

$$f(x) = e^x + x - 4$$

$f(1) = e^1 + 1 - 4$ = -0.2817 < 0

$f(2) = e^2 + 2 - 4$ = 5.389 > 0

Sing changed, therefore rood lies between 1 and 2

(b)

$$f(x) = e^x + x - 4$$

$$f'(x) = e^x + 1$$

$$x_0 = 1.2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{e^{x_n} + x_n - 4}{e^{x_n} + 1}$$

$$x_1 = 1.2 - \frac{e^{1.2} + 1.2 - 4}{e^{1.2} + 1} = 1.07961$$

$$x_2 = 1.07961 - \frac{e^{1.07961} + 1.07961 - 4}{e^{1.07961} + 1} = 1.07374$$

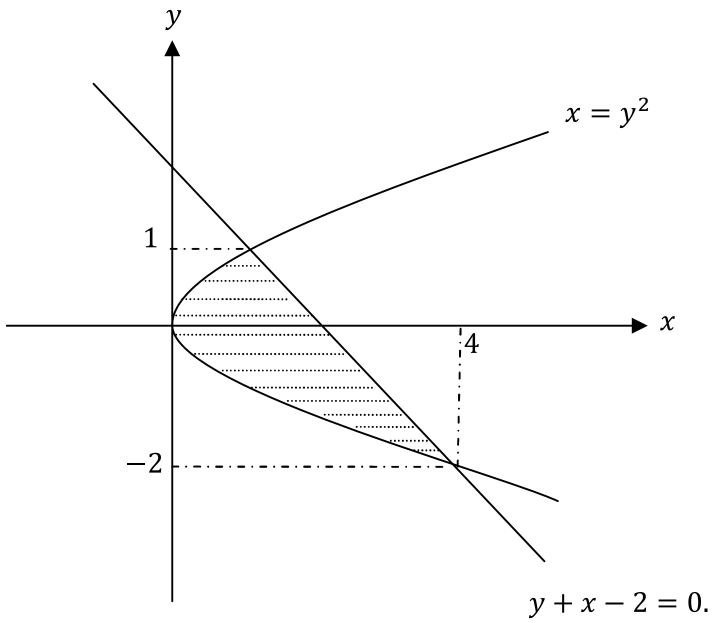
$$x_3 = 1.2 - \frac{e^{1.07374} + 1.07374 - 4}{e^{1.07374} + 1} = 1.07373$$

$$\therefore \textit{The root is } 1.0737$$

5. (a) Find the area of the region bounded by the curve $x = y^2$ and the straight line $y + x - 2 = 0$.
   (b) The region bounded by $y = x^2 + 3x, x = -3 \; and \; x = -1$ is rotated completely about the x-axis. Find the volume of the solid formed.

SOLUTION

(a)



$x = y^2$ .... (1)
$y + x - 2 = 0$ .... (2)
$From \; (2)$
$x = 2 - y$ .... (3)
Substitute (3) into (1)
$2 - y = y^2$
$y^2 + y - 2 = 0$
$(y - 1)(y + 2) = 0$
$y = 1 \; ; \quad y = -2$
$x = 1 \; ; \quad x = 4$
Intersection poins: $(1, 1) \; and \; (4, -2)$

$$Area = \left| \int_{-2}^{1} (2 - y) - y^2 \, dy \right|$$
$$= \left| \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^{1} \right|$$
$$= \left| \left[2(1) - \frac{(1)^2}{2} - \frac{(1)^3}{3} \right] - \left[2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3} \right] \right|$$
$$= \frac{9}{2} \; unit^2$$

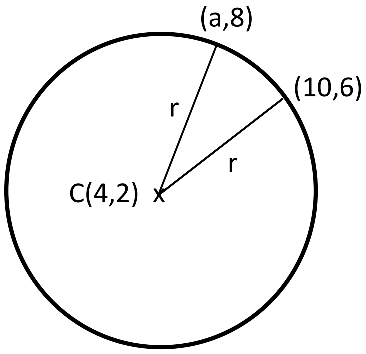
(b)

$$Volume = \pi\int_{-3}^{-1}(x^2+3x)^2dx$$
$$= \pi\int_{-3}^{-1}(x^4+6x^3+9x^2)dx$$
$$= \pi\left[\frac{x^5}{5}+\frac{6x^4}{4}+\frac{9x^3}{3}\right]_{-3}^{-1}$$
$$= \pi\left\{\left[\frac{(-1)^5}{5}+\frac{6(-1)^4}{4}+\frac{9(-1)^3}{3}\right]-\left[\frac{(-3)^5}{5}+\frac{6(-3)^4}{4}+\frac{9(-3)^3}{3}\right]\right\}$$
$$= \pi\left\{\left[\frac{-1}{5}+\frac{3}{2}-3\right]-\left[\frac{-243}{5}+\frac{243}{2}-81\right]\right\}$$
$$= \frac{32}{5}\,\pi\, unit^3$$

6. (a) A circle with center (4, -2) passes through the points (10, 6) and (a, 8). Find
   i. The value of a.
   ii. The general equation of the circle.

   (b) Find the standard equation of a parabola with its symmetric axis parrallel to the x-axis, vertex at the point (3, 2) and passing through the point (4, 4).

SOLUTION



(ai)

$$Center\ (h,k) = (4,-2)$$
$$r = \sqrt{(4-10)^2 + (-2-6)^2} = \sqrt{(a-4)^2 + (8+2)^2}$$

$$36 + 64 = (a-4)^2 + 100$$
$$(a-4)^2 = 0$$
$$a = 4$$

(aii)

$$Radius, \qquad r = \sqrt{(4-10)^2 + (-2-6)^2} = 10$$

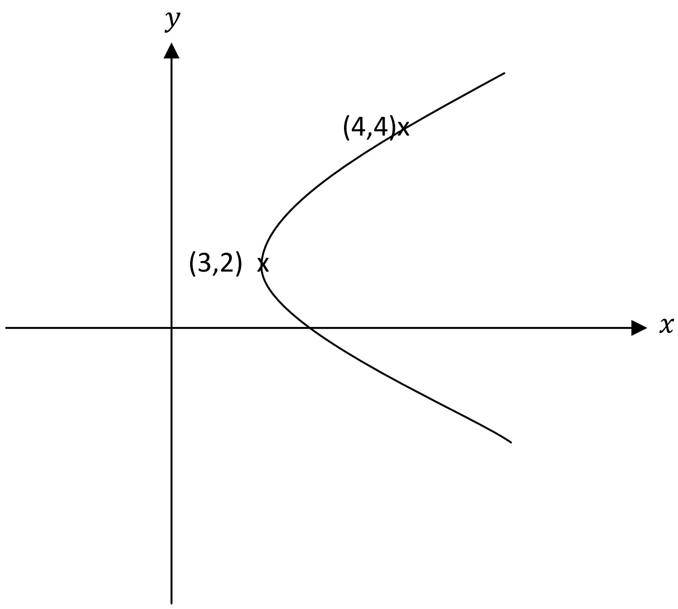
The equation of circle:

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-4)^2 + (y+2)^2 = 10^2$$
$$(x-4)^2 + (y+2)^2 = 100$$
$$x^2 + y^2 - 8x + 4y - 80 = 0$$

Standard Equation

General Equation

(b)



Vertex, (h,k) = (3,2)

Standard equation of parabola:

$$(y-k)^2 = 4p(x-h)$$
$$(y-2)^2 = 4p(x-3)$$

At (4,4):

$$(4-2)^2 = 4p(4-3)$$
$$2^2 = 4p(1)$$
$$4p = 4$$
$$p = 1$$

Standard equation of parabola:

$$(y-2)^2 = 4(1)(x-3)$$

$$(y-2)^2 = 4(x-3)$$