KEMENTERIAN PENDIDIKAN MALAYSIA

QS 025

Mid-Semester Examination

Semester II

Session 2014/2015

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1. Find the equation of the parabola with vertex (-1,3) and focus (-1,4). Sketch the graph of the parabola by labelling the vertex, focus and directrix.

2. Show that the equation $16x^2 + 4y^2 - 64x - 40y + 100 = 0$ represents an ellipse. Find the centre, foci and vertices of the ellipse.

3. Find the general solution of the linear differential equation $\frac{dy}{dx} + \frac{y}{x} = x$ by expressing y in terms of x.

4. Use the trapezoidal rule to find an approximate value for $\int_0^2 \frac{1}{1+x^2}\,dx$ with four sub-intervals.

5. By using partial fractions, show that

$$\frac{x}{x^2-1} = \frac{1}{2(x+1)} + \frac{1}{2(x-1)}$$

   Hence, evaluate $\int_2^5 \frac{x}{x^2-1}\,dx$.

6. (a) By using substitution $u = \sqrt{x-1}$, find $\int \frac{x}{\sqrt{x-1}}\,dx$.

   (b) Find $\int xe^{4x}\,dx$.

1. Find the equation of the parabola with vertex (-1,3) and focus (-1,4). Sketch the graph of the parabola by labelling the vertex, focus and directrix.

SOLUTION

v(-1,3) , f(-1,4)

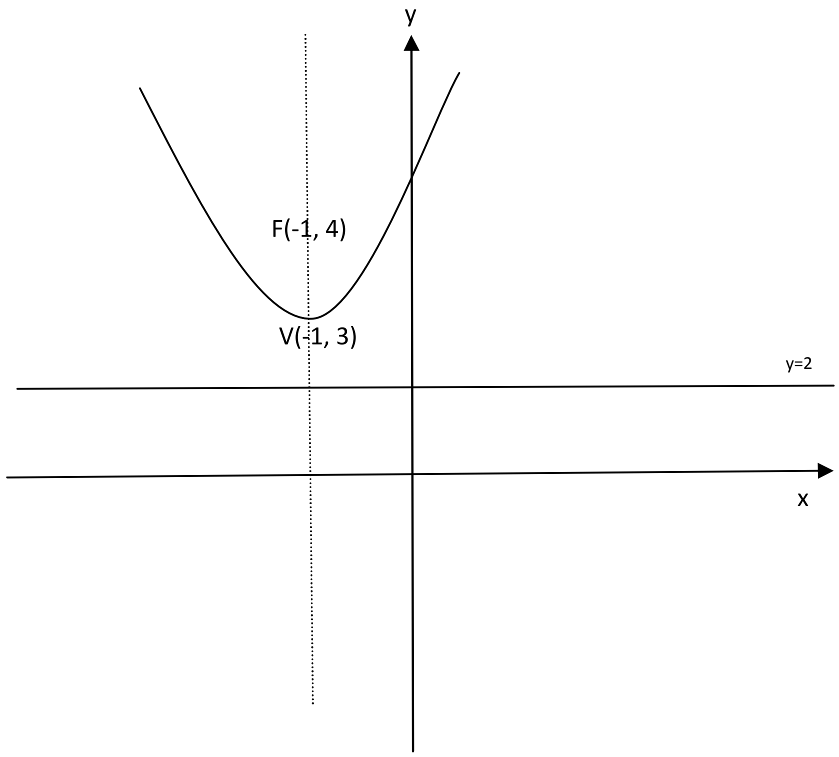
h = -1, k = 3

p = 4 – 3 = 1

$(x-h)^2 = 4p(y-k)$

$(x+1)^2 = 4(1)(y-3)$

$(x+1)^2 = 4(y-3)$



2. Show that the equation $16x^2 + 4y^2 - 64x - 40y + 100 = 0$ represents an ellipse. Find the centre, foci and vertices of the ellipse.

SOLUTION

$$16x^2 + 4y^2 - 64x - 40y + 100 = 0$$
$$16x^2 - 64x + 4y^2 - 40y = -100$$
$$16(x^2 - 4x) + 4(y^2 - 10y) = -100$$
$$16\left[x^2 - 4x + \left(\frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2\right] + 4\left[y^2 - 10y + \left(\frac{-10}{2}\right)^2 - \left(\frac{-10}{2}\right)^2\right] = -100$$
$$16[x^2 - 4x + (-2)^2 - (-2)^2] + 4[y^2 - 10y + (-5)^2 - (-5)^2] = -100$$
$$16[(x-2)^2 - 4] + 4[(y-5)^2 - 25] = -100$$
$$16(x-2)^2 - 64 + 4(y-5)^2 - 100 = -100$$
$$16(x-2)^2 + 4(y-5)^2 = -100 + 64 + 100$$
$$16(x-2)^2 + 4(y-5)^2 = 64$$
$$\frac{16(x-2)^2}{64} + \frac{4(y-5)^2}{64} = \frac{64}{64}$$
$$\frac{(x-2)^2}{4} + \frac{(y-5)^2}{16} = 1$$

Compare

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$h = 2$ $\quad$ $k = 5$

$a^2 = 4$ $\quad$ $b^2 = 16$

$a = 2$ $\quad$ $b = 4$

$$c^2 = b^2 - a^2$$
$$c^2 = 16 - 4$$
$$c = \sqrt{12}$$

Centre: (h, k) =(2, 5)

$V_1(h, k+b) = V_1(2, 5+4) = V_1(2, 9)$

$V_2(h, k-b) = V_2(2, 5-4) = V_1(2, 1)$

$F_1(h, k+c) = F_1(2, 5+\sqrt{12})$

$F_2(h, k-c) = F_2(2, 5-\sqrt{12})$

3. Find the general solution of the linear differential equation $\frac{dy}{dx}+\frac{y}{x}=x$ by expressing y in terms of x.

SOLUTION

$$\frac{dy}{dx}+\frac{1}{x}(y)=x$$
$$\frac{dy}{dx}+P(x)y=Q(x)$$

$$\frac{dy}{dx}+\frac{y}{x}=x$$
$$\frac{dy}{dx}+\frac{1}{x}(y)=x$$
$$P(x)=\frac{1}{x},\quad Q(x)=x$$
$$V(x)=e^{\int P(x)dx}$$
$$V(x)=e^{\int\frac{1}{x}dx}$$
$$V(x)=e^{\ln(x)}$$
$$V(x)=x$$

$$V(x)y=\int V(x)Q(x)dx$$

$$xy=\int x(x)dx$$
$$xy=\int x^2dx$$
$$xy=\frac{x^3}{3}+c$$
$$y=\frac{x^3}{3x}+\frac{c}{x}$$
$$y=\frac{x^2}{3}+\frac{c}{x}$$

4. Use the trapezoidal rule to find an approximate value for $\int_0^2 \frac{1}{1+x^2}\,dx$ with four sub-intervals.

SOLUTION

$$\int_0^2 \frac{1}{1+x^2}\,dx$$

$$a = 0,\ \ b\ = 2,\ \ n = 4,\ \ f(x) = \frac{1}{1+x^2}$$

$$h = \frac{b-a}{n}\ = \frac{2-0}{4} = 0.5$$

X	$f(x) = \frac{1}{1+x^2}$	
0.0	1.0	
0.5		0.8000
1.0		0.5000
1.5		0.3077
2.0	0.2	
Total	1.2	1.6077

$$\int_0^2 \frac{1}{1+x^2}\,dx = \frac{h}{2}[(y_0 + y_1) + 2(y_1 + y_2 + y_3)]$$

$$\int_0^2 \frac{1}{1+x^2}\,dx = \frac{0.5}{2}[(1.2) + 2(1.6077)]$$

$$\int_0^2 \frac{1}{1+x^2}\,dx = 1.1039$$

5. By using partial fractions, show that

$$\frac{x}{x^2-1}=\frac{1}{2(x+1)}+\frac{1}{2(x-1)}$$

Hence, evaluate $\int_2^5 \frac{x}{x^2-1}dx$.

SOLUTION

$$\frac{x}{(x+1)(x-1)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}$$

$$\frac{x}{(x+1)(x-1)}=\frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$x=A(x-1)+B(x+1)$$

$When\ x=1:$

$$1=A(1-1)+B(1+1)$$
$$2B=1$$
$$B=\frac{1}{2}$$

$When\ x=-1:$

$$-1=A(-1-1)+B(-1+1)$$
$$-2A=-1$$
$$A=\frac{1}{2}$$

$$\therefore \frac{x}{x^2-1}=\frac{1}{2(x+1)}+\frac{1}{2(x-1)}$$

$$\int_2^5 \frac{x}{x^2-1}dx=\int_2^5 \frac{1}{2(x+1)}+\frac{1}{2(x-1)}dx$$

$$\int_2^5 \frac{x}{x^2-1}dx=\frac{1}{2}\int_2^5 \frac{1}{(x+1)}+\frac{1}{(x-1)}dx$$

$$=\frac{1}{2}[\ln(x+1)+\ln(x-1)]_2^5$$

$$= \frac{1}{2}[\ln(x+1)(x-1)]_2^5$$

$$= \frac{1}{2}[\ln(5+1)(5-1) - \ln(2+1)(2-1)]$$

$$= \frac{1}{2}[\ln(6)(4) - \ln(3)(1)]$$

$$= \frac{1}{2}[\ln 24 - \ln 3]$$

$$= \frac{1}{2}\left[\ln\frac{24}{3}\right]$$

$$= \frac{1}{2}[\ln 8]$$

$$= 1.040$$

6. (a) By using substitution $u = \sqrt{x-1}$, find $\int \frac{x}{\sqrt{x-1}}dx$.

(b) Find $\int xe^{4x}dx$.

SOLUTION

6(a)

$$\int \frac{x}{\sqrt{x-1}}dx$$

$u = \sqrt{x-1}$ ➔ $u^2 = x - 1$ ➔ $x = u^2 + 1$

$$u = (x-1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x-1}}$$

$$\frac{du}{dx} = \frac{1}{2u}$$

$$dx = 2udu$$

$$\int \frac{x}{\sqrt{x-1}}dx = \int \frac{u^2+1}{u}(2udu)$$

$$= 2\int u^2 + 1\,du$$

$$= 2\left[\frac{u^3}{3} + u\right] + c$$

$$= \frac{2u^3}{3} + 2u + c$$

$$= \frac{2\left(\sqrt{x-1}\right)^3}{3} + 2\sqrt{x-1} + c$$

6(b)

$$\int xe^{4x}dx$$

$u = x$ $\quad$ $dv = e^{4x}dx$

$du = dx$ $\quad$ $v = \int e^{4x}dx$

$v = \frac{e^{4x}}{4}$

$$\int xe^{4x}dx = \frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4}dx$$

$$\int xe^{4x}dx = \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + c$$